# VII. Elements of the theory of distributions

## VII.1 Space of test functions and weak derivatives

**Notation** (reminder from Chapter IV):

- $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \dots\}$
- Elements of $\mathbb{N}_0^d$ are called **multiindices**. For $\alpha \in \mathbb{N}_0^d$ we set $|\alpha| = \alpha_1 + \dots + \alpha_d$. This number is called **the order of the multiindex** $\alpha$.
- If $\Omega \subset \mathbb{R}^d$ is an open set, $f \in \mathcal{C}^\infty(\Omega, \mathbb{F})$ and $\boldsymbol{a} \in \Omega$, we set
$$D^\alpha f(\boldsymbol{a}) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}(\boldsymbol{a}).$$

**Definition.** Let $d \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^d$ be an open set.

(a) If $f : \Omega \to \mathbb{F}$ is continuous, its **support** is the set
$$\operatorname{spt} f = \overline{\{\boldsymbol{x} \in \Omega; f(\boldsymbol{x}) \neq 0\}},$$
where the closure is taken in $\Omega$.

(b) Let
$$\mathscr{D}(\Omega, \mathbb{F}) = \{f \in \mathcal{C}^\infty(\Omega, \mathbb{F}); \operatorname{spt} f \text{ is a compact subset } \Omega\}.$$
Elements of $\mathscr{D}(\Omega, \mathbb{F})$ are called **test functions**, the space $\mathscr{D}(\Omega, \mathbb{F})$ is called **the space of test functions**.

(c) A measurable function $f : \Omega \to \mathbb{F}$ is called **locally integrable in** $\Omega$, if for any $\boldsymbol{x} \in \Omega$ there exists $r > 0$ such that $f$ is Lebesgue integrable on $U(\boldsymbol{x}, r)$ (i.e., $\int_{U(\boldsymbol{x},r)} |f| < \infty$). The space of all locally integrable functions in $\Omega$ is denoted by $L^1_{\text{loc}}(\Omega, \mathbb{F})$. (More precisely, it is the space of all equivalence classes, where we identify functions equal almost everywhere.)

(d) Choose a non-negative $h \in \mathscr{D}(\mathbb{R}^d)$ such that $\operatorname{spt} \varphi \subset U(0, 1)$ and $\int_{\mathbb{R}^d} h = 1$. For $j \in \mathbb{N}$ we define a function $h_j$ by
$$h_j(\boldsymbol{x}) = j^d h(j\boldsymbol{x}) \text{ for } \boldsymbol{x} \in \mathbb{R}^d.$$
The sequence $(h_j)$, obtained in this way is called an **approximate unit** in $\mathscr{D}(\mathbb{R}^d)$ or a **smoothing kernel**.

**Lemma 1.** *Let $\Omega \subset \mathbb{R}^d$ be open. Then $\mathscr{D}(\Omega)$ is a dense subspace of $L^p(\Omega)$ for any $p \in [1, \infty)$.*

**Lemma 2.** *Let $\Omega \subset \mathbb{R}^d$ be an open set.*

- *Let $\mu$ be a (finite) signed or complex regular Borel measure on $\Omega$. If $\int_\Omega \varphi \, d\mu = 0$ for any $\varphi \in \mathscr{D}(\Omega)$, then $\mu = 0$.*
- *Let $f \in L^1_{\text{loc}}(\Omega)$ and $\int_\Omega f\varphi = 0$ for any $\varphi \in \mathscr{D}(\Omega)$. Then $f = 0$ almost everywhere on $\Omega$.*

**Definition.** Let $(a,b) \subset \mathbb{R}$ be an open interval an let $f \in L^1_{\text{loc}}((a,b))$.

- A function $g \in L^1_{\text{loc}}((a,b))$ is called a **weak derivative** of a function $f$, if for any $\varphi \in \mathscr{D}((a,b))$ we have
$$\int_a^b f\varphi' = -\int_a^b g\varphi.$$
- Let $\mu$ be a finite regular Borel measure on $(a,b)$ (signed or complex). The measure $\mu$ is said to be a **weak derivative** of a function $f$, if for any $\varphi \in \mathscr{D}((a,b))$ we have
$$\int_a^b f\varphi' = -\int_{(a,b)} \varphi \,\mathrm{d}\mu.$$

**Proposition 3.** *Let $(a,b) \subset \mathbb{R}$ be an open interval and let $f \in L^1_{\text{loc}}((a,b))$. If $\int_a^b f\varphi' = 0$ for any $\varphi \in \mathscr{D}((a,b))$, the function $f$ is constant (i.e., there exists a constant $c$ such that $f = c$ almost everywhere on $(a,b)$).*

*In other words: If the zero function is a weak derivative of a function $f \in L^1_{\text{loc}}((a,b))$, the function $f$ is constant (in the above-mentioned sense).*

**Theorem 4.** *Let $f \in L^1_{\text{loc}}((a,b))$.*

(a) *The weak derivative of $f$ is uniquely determined. I.e., if two functions $g_1, g_2 \in L^1_{\text{loc}}((a,b))$ are weak derivatives of a function $f$, then $g_1 = g_2$ almost everywhere. Similarly, if two measures $\mu_1, \mu_2$ are weak derivatives of a function $f$, then $\mu_1 = \mu_2$.*

(b) *If $f$ is absolutely continuous on $[a,b]$, it has a finite derivative almost everywhere, $f' \in L^1((a,b))$ and $f'$ is the weak derivative of $f$.*
*Conversely, if a function $f$ has a weak derivative $g \in L^1((a,b))$, there exists a function $f_0$ absolutely continuous on $[a,b]$, equal to $f$ almost everywhere on $(a,b)$. In this case $g = f_0'$ almost everywhere.*
*More generally, a function $f$ has a weak derivative in $L^1_{\text{loc}}((a,b))$ if and only if there exists function $f_0$ locally absolutely continuous on $(a,b)$ (i.e., absolutely continuous on each closed subinterval $[c,d] \subset (a,b)$) such that $f_0 = f$ almost everywhere.*

(c) *There exists a finite measure $\mu$, which is a weak derivative of function $f$ if and only if there exists a function $f_0$ of bounded variation on $[a,b]$ such that $f_0 = f$ almost everywhere on $(a,b)$. In this case for each subinterval $(c,d) \subset (a,b)$ we have*
$$\mu((c,d)) = \lim_{x\to d-} f_0(x) - \lim_{x\to c+} f_0(x).$$
*Moreover, $\mu$ is real-valued if and only if $f_0$ may be real-valued and $\mu$ is non-negative if and only if $f_0$ may be non-increasing.*