VII.2 Distributions – basic properties and operations

Definition. Let $\Omega \subset \mathbb{R}^d$ be an open set, (φ_n) a sequence in $\mathscr{D}(\Omega)$ and $\varphi \in \mathscr{D}(\Omega)$. We say that the sequence (φ_n) **converges to φ in $\mathscr{D}(\Omega)$**, if the following two conditions are fulfilled:

- There exists $K \subset \Omega$ compact such that $\operatorname{spt} \varphi_n \subset K$ for each $n \in \mathbb{N}$.
- $D^\alpha \varphi_n \rightrightarrows D^\alpha \varphi$ on K for each multiindex $\alpha \in \mathbb{N}_0^d$.

This is expressed by writing '$\varphi_n \to \varphi$ in $\mathscr{D}(\Omega)$'.

Remark. Let $\alpha \in \mathbb{N}_0^d$ be a multiindex.

- If $\varphi \in \mathscr{D}(\Omega)$, then $D^\alpha \varphi \in \mathscr{D}(\Omega)$.
- If $\varphi_n \to \varphi$ in $\mathscr{D}(\Omega)$, then $D^\alpha \varphi_n \to D^\alpha \varphi$ in $\mathscr{D}(\Omega)$.

Notation: Let $\Omega \subset \mathbb{R}^d$ be an open set.

- For $\varphi \in \mathscr{D}(\Omega)$ and $N \in \mathbb{N}_0$ we define

$$\|\varphi\|_N = \max\{\|D^\alpha \varphi\|_\infty \,;\, \alpha \in \mathbb{N}_0^d, |\alpha| \le N\} = \sup\{|D^\alpha \varphi(x)| \,;\, x \in \Omega, \alpha \in \mathbb{N}_0^d, |\alpha| \le N\}.$$

- If $K \subset \Omega$ is a compact subset, we set

$$\mathscr{D}_K(\Omega) = \{\varphi \in \mathscr{D}(\Omega); \operatorname{spt} \varphi \subset K\}.$$

Lemma 5. *Let $\Omega \subset \mathbb{R}^d$ be an open set.*

(a) $\|\cdot\|_N$ *is a norm on $\mathscr{D}(\Omega)$ for each $N \in \mathbb{N}_0$.*
(b) *If $K \subset \Omega$ is a compact subset, then the space $\mathscr{D}_K(\Omega)$ rquipped with the sequence of norms $(\|\cdot\|_N)$ is a Fréchet space.*

Proposition 6. *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\Lambda : \mathscr{D}(\Omega) \to \mathbb{F}$ be a linear functional. The following conditions are equivalent:*

(1) $\forall (\varphi_n) \subset \mathscr{D}(\Omega)\ \forall \varphi \in \mathscr{D}(\Omega) : \varphi_n \to \varphi$ *in* $\mathscr{D}(\Omega) \Rightarrow \Lambda(\varphi_n) \to \Lambda(\varphi)$.
(2) $\forall (\varphi_n) \subset \mathscr{D}(\Omega)\ \varphi_n \to 0$ *v* $\mathscr{D}(\Omega) \Rightarrow \Lambda(\varphi_n) \to 0$.
(3) *For each $K \subset \Omega$ compact the restriction $\Lambda|_{\mathscr{D}_K(\Omega)}$ is continuous on $\mathscr{D}_K(\Omega)$.*
(4) *For each compact subset $K \subset \Omega$ there exist $N \in \mathbb{N}_0$ and $C > 0$ such that*
$$|\Lambda(\varphi)| \le C \|\varphi\|_N\,, \quad \varphi \in \mathscr{D}_K(\Omega).$$

Definition. Let $\Omega \subset \mathbb{R}^d$ be an open set.

- By a **distribution** on Ω we mean a linear functional $\Lambda : \mathscr{D}(\Omega) \to \mathbb{F}$ satisfying the equivalent conditions from Proposition 6.
- The space of all distributions on Ω is denoted by $\mathscr{D}'(\Omega)$.
- A distribution Λ on Ω is said to be of **finite order**, if in condition (3) form Proposition 6 the number $N \in \mathbb{N}_0$ may be chosen indepent on K. The smallest such N is called **the order of the distribution Λ**.

Examples 7. *Let $\Omega \subset \mathbb{R}^d$ be an open set.*

(1) *For $f \in L^1_{\mathrm{loc}}(\Omega)$ we define*
$$\Lambda_f(\varphi) = \int_\Omega f\varphi, \quad \varphi \in \mathscr{D}(\Omega).$$
Then Λ_f is a distribution of order 0 on Ω. It is called **the regular distribution induced by f**.

(2) *If μ is a nonnegative regular Borel measure on Ω which is finite on compact subsets of Ω, then*

$$\Lambda_\mu(\varphi) = \int_\Omega \varphi \,\mathrm{d}\mu, \quad \varphi \in \mathscr{D}(\Omega),$$

is a distribution of order 0 on Ω.

(3) *If μ is a finite signed or complex regular Borel measure on Ω, the mapping Λ_μ defined by the same formula as in the previous item is a distribution of order 0 on Ω.*

(4) *The mapping*

$$\Lambda(\varphi) = \varphi'(0), \qquad \varphi \in \mathscr{D}(\mathbb{R}),$$

is a distribution of order 1 on $\mathbb{R}$. This distribution is not of the form Λ_f or Λ_μ from the preceding items.

(5) *The mapping*

$$\Lambda(\varphi) = \sum_{n=1}^{\infty} \varphi^{(n)}(n), \qquad \varphi \in \mathscr{D}(\mathbb{R}),$$

is a distribution on $\mathbb{R}$, which is not of finite order.

Remarks. Lemma 2 implies the following assertions:

- If $f, g \in L^1_{\mathrm{loc}}(\Omega)$ are such that $\Lambda_f = \Lambda_g$, then $f = g$ almost everywhere on Ω. This explains why distributions are sometimes called **generalized functions**.
- If μ and ν are two measures satisfying $\Lambda_\mu = \Lambda_\nu$, necessarily $\mu = \nu$.
- If $f \in L^1_{\mathrm{loc}}(\Omega)$ and μ is a measure such that $\Lambda_f = \Lambda_\mu$, then $\mu(A) = \int_A f \,\mathrm{d}\lambda^d$ for any $A \subset \Omega$ Borel.

Definition. Let $\Omega \subset \mathbb{R}^d$ be an open set and let Λ be a distribution on Ω.

- If $\alpha \in \mathbb{N}_0^d$ is a multiindex, then the **α-th derivative of the distribution Λ** is the mapping $D^\alpha \Lambda$ defined by the formula
$$D^\alpha \Lambda(\varphi) = (-1)^{|\alpha|} \Lambda(D^\alpha \varphi), \quad \varphi \in \mathscr{D}(\Omega).$$
- If $f \in \mathcal{C}^\infty(\Omega)$, **the multiple of the distribution Λ by the function f** is the mapping $f\Lambda$ defined by the formula
$$(f\Lambda)(\varphi) = \Lambda(f\varphi), \quad \varphi \in \mathscr{D}(\Omega).$$

Remark. If $d = 1$, we write Λ' in place of $D^1\Lambda$, Λ'' in place of $D^2\Lambda$, in general $\Lambda^{(n)}$ in place of $D^n\Lambda$.

Proposition 8. *Let $\Omega \subset \mathbb{R}^d$ be an open set. Then:*

(a) *For each $\Lambda \in \mathscr{D}'(\Omega)$ and each multiindex $\alpha \in \mathbb{N}_0^d$ the mapping $D^\alpha\Lambda$ is also a distribution on Ω.*

(b) *For each $f \in \mathcal{C}^\infty(\Omega)$ we have $D^\alpha \Lambda_f = \Lambda_{D^\alpha f}$.*

(c) *If $d = 1$, $\Omega = (a, b)$ and $f \in L^1_{\mathrm{loc}}((a, b))$, then*
   - *$(\Lambda_f)' = \Lambda_g$ (where $g \in L^1_{\mathrm{loc}}((a, b))$) if and only if g is the weak derivative of f;*
   - *$(\Lambda_f)' = \Lambda_\mu$ (where μ is a finite measure) if and only if μ is the weak derivative of f.*

(d) *If $\Lambda \in \mathscr{D}'(\Omega)$ and $f \in \mathcal{C}^\infty(\Omega)$, then $f\Lambda$ is a distribution on Ω.*

(e) *If $f \in \mathcal{C}^\infty(\Omega)$ and $g \in L^1_{\mathrm{loc}}(\Omega)$, then $f\Lambda_g = \Lambda_{fg}$.*

Proposition 9.

(a) *Let $\Lambda \in \mathscr{D}'((a, b))$ satisfy $\Lambda' = 0$. Then there exists $c \in \mathbb{F}$ such that $\Lambda = \Lambda_c$.*

(b) *More generally, if $\Omega \subset \mathbb{R}^d$ is an open connected set and $\Lambda \in \mathscr{D}'(\Omega)$ is such that $D^\alpha\Lambda = 0$ for each multiindex α satisfying $|\alpha| = 1$, then there exists $c \in \mathbb{F}$ such that $\Lambda = \Lambda_c$.*