VII.4 Convolution of distributions

Notation. Let $M \subset \mathbb{R}^d$ and let $f : M \to \mathbb{F}$ be a function.

- For $\boldsymbol{y} \in \mathbb{R}^d$ we define a function $\tau_{\boldsymbol{y}} f$ (a **translate** of the function f) by the formula
$$(\tau_{\boldsymbol{y}} f)(\boldsymbol{x}) = f(\boldsymbol{x} - \boldsymbol{y}), \quad \boldsymbol{x} \in \boldsymbol{y} + M.$$
- By the **reflexion** of f we mean the function $\check{f}$ defined by the formula
$$\check{f}(\boldsymbol{x}) = f(-\boldsymbol{x}), \quad \boldsymbol{x} \in -M.$$
- For $\boldsymbol{a}, \boldsymbol{e} \in \mathbb{R}^d$ we define $\partial_{\boldsymbol{e}} f(\boldsymbol{a}) = \lim\limits_{r \to 0} \dfrac{f(\boldsymbol{a} + r\boldsymbol{e}) - f(\boldsymbol{a})}{r}$, if the limit exists. (If $\boldsymbol{e}$ is a unit vector, it is a directional derivative of f at $\boldsymbol{a}$ in the direction $\boldsymbol{e}$.)

Lemma 13. *Let $\varphi \in \mathscr{D}(\mathbb{R}^d)$.*

(a) *If $\boldsymbol{x}_n \to \boldsymbol{x}$ in $\mathbb{R}^d$, then $\tau_{\boldsymbol{x}_n} \varphi \to \tau_{\boldsymbol{x}} \varphi$ v $\mathscr{D}(\mathbb{R}^d)$.*

(b) *Let $\boldsymbol{e} \in \mathbb{R}^d$. Then $\partial_{\boldsymbol{e}} \varphi \in \mathscr{D}(\mathbb{R}^d)$. Moreover, if we define the function φ_r for $r \in \mathbb{R} \setminus \{0\}$ by the formula*
$$\varphi_r(\boldsymbol{x}) = \tfrac{1}{r}(\varphi(\boldsymbol{x} + r\boldsymbol{e}) - \varphi(\boldsymbol{x})), \quad \boldsymbol{x} \in \mathbb{R}^d,$$
i.e., $\varphi_r = \frac{1}{r}(\tau_{-r\boldsymbol{e}} \varphi - \varphi)$, then $\varphi_r \to \partial_{\boldsymbol{e}} \varphi$ in $\mathscr{D}(\mathbb{R}^d)$ for $r \to 0$.

Proposition 14. *Let $d_1, d_2 \in \mathbb{N}$ and let $\varphi \in \mathscr{D}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$.*

(a) *Let $\Lambda \in \mathscr{D}'(\mathbb{R}^{d_1})$. For $\boldsymbol{y} \in \mathbb{R}^{d_2}$ we define $\psi(\boldsymbol{y}) = \Lambda(\boldsymbol{x} \mapsto \varphi(\boldsymbol{x}, \boldsymbol{y}))$. Then $\psi \in \mathscr{D}(\mathbb{R}^{d_2})$ and for each multiindex $\alpha \in \mathbb{N}_0^{d_2}$ we have $D^\alpha \psi(\boldsymbol{y}) = \Lambda(\boldsymbol{x} \mapsto D^{(\boldsymbol{o}, \alpha)} \varphi(\boldsymbol{x}, \boldsymbol{y}))$ for $\boldsymbol{y} \in \mathbb{R}^{d_2}$.*

(b) (Fubini Theorem for distributions) *Let $\Lambda_1 \in \mathscr{D}'(\mathbb{R}^{d_1})$ and $\Lambda_2 \in \mathscr{D}'(\mathbb{R}^{d_2})$. Then*
$$\Lambda_2(\boldsymbol{y} \mapsto \Lambda_1(\boldsymbol{x} \mapsto \varphi(\boldsymbol{x}, \boldsymbol{y}))) = \Lambda_1(\boldsymbol{x} \mapsto \Lambda_2(\boldsymbol{y} \mapsto \varphi(\boldsymbol{x}, \boldsymbol{y}))).$$

Definition. Let U be a distribution on $\mathbb{R}^d$ and let $\varphi \in \mathscr{D}(\mathbb{R}^d)$. The **convolution of φ and U** is the function $U * \varphi$ defined by the formula
$$U * \varphi(\boldsymbol{x}) = U(\tau_{\boldsymbol{x}} \check{\varphi}) = U(\boldsymbol{y} \mapsto \varphi(\boldsymbol{x} - \boldsymbol{y})), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

Remark: Since $\tau_{\boldsymbol{x}} \check{\varphi} \in \mathscr{D}(\mathbb{R}^d)$ whenever $\boldsymbol{x} \in \mathbb{R}^d$ and $\varphi \in \mathscr{D}(\mathbb{R}^d)$, $U * \varphi$ is a well-defined function on $\mathbb{R}^d$.

Theorem 15 (on the convolution of a distribution and a test function). *Let U be a distribution on $\mathbb{R}^d$ and let $\varphi, \psi \in \mathscr{D}(\mathbb{R}^d)$. Then:*

(a) *If $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, then $\Lambda_f * \varphi = f * \varphi$.*

(b) *$U * \varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ and for each multiindex α we have*
$$D^\alpha(U * \varphi) = (D^\alpha U) * \varphi = U * D^\alpha \varphi.$$

(c) $\operatorname{spt}(U * \varphi) \subset \operatorname{spt} U + \operatorname{spt} \varphi$. *In particular, if U has compact support, then $U * \varphi \in \mathscr{D}(\mathbb{R}^d)$.*

(d) *If (h_j) is an approximate unit in $\mathscr{D}(\mathbb{R}^d)$, then $\Lambda_{U * h_j} \to U$ in $\mathscr{D}'(\mathbb{R}^d)$.*

(e) *For each $\boldsymbol{x} \in \mathbb{R}^d$ we have $\tau_{\boldsymbol{x}}(U * \varphi) = (\tau_{\boldsymbol{x}} U) * \varphi = U * \tau_{\boldsymbol{x}} \varphi$.*

(f) *$U * (\varphi * \psi) = (U * \varphi) * \psi$.*

Notation. If U is a distribution on $\mathbb{R}^d$,

- its **translate by $\boldsymbol{x} \in \mathbb{R}^d$** is the mapping $\tau_{\boldsymbol{x}} U$ defined by the formula
$$\tau_{\boldsymbol{x}} U(\varphi) = U(\tau_{-\boldsymbol{x}} \varphi) = U(\boldsymbol{y} \mapsto \varphi(\boldsymbol{y} + \boldsymbol{x})), \quad \varphi \in \mathscr{D}(\mathbb{R}^d);$$
- its **reflexion** is the mapping $\check{U}$ defined by the formula
$$\check{U}(\varphi) = U(\check{\varphi}) = U(\boldsymbol{y} \mapsto \varphi(-\boldsymbol{y})), \quad \varphi \in \mathscr{D}(\mathbb{R}^d).$$

Remark. If U is a distribution on $\mathbb{R}^d$ and $\boldsymbol{x} \in \mathbb{R}^d$, then $\tau_{\boldsymbol{x}} U$ and $\check{U}$ are also distributions on $\mathbb{R}^d$. If $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, then $\tau_{\boldsymbol{x}} \Lambda_f = \Lambda_{\tau_{\boldsymbol{x}} f}$ a $\overset{\vee}{\Lambda}_f = \Lambda_{\check{f}}$.

Remark on the convolution of two distributions: Let U and V be two distributions on $\mathbb{R}^d$. It would be natural to define their convolution by the formula

$$(U * V)(\varphi) = U(\check{V} * \varphi), \quad \varphi \in \mathscr{D}(\mathbb{R}^d).$$

Indeed, if $V = \Lambda_\psi$ for some $\psi \in \mathscr{D}(\mathbb{R}^d)$, then we deduce by Proposition 14(b) that

$$\Lambda_{U*\psi}(\varphi) = U(\check{\psi} * \varphi), \quad \varphi \in \mathscr{D}(\mathbb{R}^d).$$

However, it is not possible to define convolution in such generality, because $\check{V} * \varphi$ need not belong to $\mathscr{D}(\mathbb{R}^d)$ (it is a C^∞ function, but its support need not be compact). Nonetheless in some special cases the above formula provides a meaningful definition. It may be caused mainly by one of the following reasons:

- It may follow from the special properties of V that $\check{V} * \varphi$ must have compact support.
- It may follow from the special properties of U that U may be naturally extended to the space of all C^∞ functions.
- Combination of these two possibilities: It may follow from the special properties of U and V that U may be naturally extended to a larger space and $\check{V} * \varphi$ must belong to that space.

Examples of situations when the convolution $U * V$ is well defined:

(1) V has compact support: Then $\check{V} * \varphi$ has compact support for any $\varphi \in \mathscr{D}(\mathbb{R}^d)$ (see Theorem 15(c)).

(2) U has compact support: We choose a nonnegative function $\psi \in \mathscr{D}(\mathbb{R}^d)$ constantly equal to 1 on an open set containing $\operatorname{spt} U$. We then extend U to $C^\infty(\mathbb{R}^d)$ by $\widetilde{U}(f) = U(\psi f)$, $f \in C^\infty(\mathbb{R}^d)$.

(3) $(\overline{U(\boldsymbol{o}, r)} - \operatorname{spt} V) \cap \operatorname{spt} U$ is compact for each $r > 0$: For $r > 0$ we choose a nonnegative function $\psi_r \in \mathscr{D}(\mathbb{R}^d)$ constantly equal to 1 on an open set containing $(\overline{U(\boldsymbol{o}, r)} - \operatorname{spt} V) \cap \operatorname{spt} U$. Then we extend U to the space

$$\{f \in C^\infty(\mathbb{R}^d); \exists r > 0 \colon \operatorname{spt} f \subset \overline{U(\boldsymbol{o}, r)} - \operatorname{spt} V\}$$

by the formula

$$\widetilde{U}(f) = U(\psi_r f), \quad \operatorname{spt} f \subset \overline{U(\boldsymbol{o}, r)} - \operatorname{spt} V.$$

(4) Assume there exist $m, n \in \mathbb{N}_0$ and $c, d > 0$ such that

$$|U(\varphi)| \le c \|\varphi\|_n \text{ and } |V(\varphi)| \le d \|\varphi\|_m \text{ for } \varphi \in \mathscr{D}(\mathbb{R}^d).$$

Then U may be canonically extended to the space

$$\{f \in C^\infty(\mathbb{R}^d); \|f\|_n < \infty\}$$

and for each $\varphi \in \mathscr{D}(\mathbb{R}^d)$ we have $\left\|\check{V} * \varphi\right\|_n \le d \|\varphi\|_{m+n}$. Hence $U * V$ may be defined in a natural way and the resulting distribution satisfies

$$|(U * V)(\varphi)| \le cd \|\varphi\|_{m+n} \text{ for } \varphi \in \mathscr{D}(\mathbb{R}^d).$$

Proposition 16. *Let U and V be distributions on $\mathbb{R}^d$ satisfying one of the above versions of assumptions. Then:*

(a) *$U * V$ is a distribution on $\mathbb{R}^d$ and $U * V = V * U$.*

(b) *If $V = \Lambda_\varphi$ for some $\varphi \in \mathscr{D}(\mathbb{R}^d)$, then $U * V = \Lambda_{U*\varphi}$.*

(c) *If $U = \Lambda_f$ and $V = \Lambda_\varphi$ where $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\varphi \in \mathscr{D}(\mathbb{R}^d)$, then $U * V = \Lambda_{f*\varphi}$.*

(d) *If $U = \Lambda_f$ and $V = \Lambda_g$, kde $f, g \in L^1(\mathbb{R}^d)$, then $U * V = \Lambda_{f*g}$.*

(e) *$\operatorname{spt}(U * V) \subset \operatorname{spt} U + \operatorname{spt} V$. In particular, if both U and V have compact support, $U * V$ has compact support as well.*

(f) *For each multiindex α we have*

$$D^\alpha (U * V) = (D^\alpha U) * V = U * D^\alpha V.$$

(g) *$U = U * \Lambda_{\delta_0}$ and $D^\alpha U = U * D^\alpha \Lambda_{\delta_0}$ for each multiindex α.*