VII.6 Convolutions and the Fourier transform of tempered distributions

Notation and convention: Recall that λ^d denotes the Lebesgue measure on $\mathbb{R}^d$. Set $m_d = (2\pi)^{-d/2}\lambda^d$. In this section $L^p(\mathbb{R}^d)$ will denote the space $L^p(m_d)$. The convolution will be considered with respect to this measure as well, i.e.,

$$f * g(\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{y})g(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}m_d(\boldsymbol{y}) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(\boldsymbol{y})g(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$

Moreover, all spaces are assumed to be complex.

Reminder:

- If $f \in L^1(\mathbb{R}^d)$, its Fourier transform is defined by the formula

$$\widehat{f}(\boldsymbol{t}) = \int_{\mathbb{R}^d} f(\boldsymbol{x})e^{-i\langle \boldsymbol{t},\boldsymbol{x}\rangle}\,\mathrm{d}m_d(\boldsymbol{x}) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(\boldsymbol{x})e^{-i\langle \boldsymbol{t},\boldsymbol{x}\rangle}\,\mathrm{d}\boldsymbol{x}, \quad \boldsymbol{t}\in\mathbb{R}^d.$$

- The Fourier transform maps $L^1(\mathbb{R}^d)$ to $C_0(\mathbb{R}^d)$.
- The Fourier transform maps $\mathscr{S}$ onto $\mathscr{S}$ and $\widehat{\widehat{f}} = \check{f}$ for $f \in \mathscr{S}$.

Lemma 24. *The Fourier transform is an isomorphism of $\mathscr{S}$ onto $\mathscr{S}$.*

Definition. Let Λ be a tempered distribution on $\mathbb{R}^d$. By its **Fourier transform** we mean the mapping

$$\widehat{\Lambda}(\varphi) = \Lambda(\widehat{\varphi}), \quad \varphi \in \mathscr{S}.$$

Reminder:

- If P is a polynomial on $\mathbb{R}^d$, by $\check{P}$ we denote the polynomial on $\mathbb{R}^d$ defined by the formula $\check{P}(\boldsymbol{t}) = P(i\boldsymbol{t})$ for $\boldsymbol{t}\in\mathbb{R}^d$. If P is of the form

$$P(\boldsymbol{t}) = \sum_{\alpha\in\mathbb{N}_0^d, |\alpha|\le N} c_\alpha \boldsymbol{t}^\alpha, \quad \boldsymbol{t}\in\mathbb{R}^d,$$

  where $N\in\mathbb{N}_0$ and c_α, $\alpha\in\mathbb{N}_0^d$, $|\alpha|\le N$ are some complex numbers, then

$$\check{P}(\boldsymbol{t}) = \sum_{\alpha\in\mathbb{N}_0^d, |\alpha|\le N} i^{|\alpha|} c_\alpha \boldsymbol{t}^\alpha, \quad \boldsymbol{t}\in\mathbb{R}^d.$$

- If P is a polynomial on $\mathbb{R}^d$ of the above form and f is a $\mathcal{C}^\infty$ function on $\mathbb{R}^d$ (or, more generally, on an open subset of $\mathbb{R}^d$), by $P(D)f$ we denote the function defined by the formula

$$P(D)f = \sum_{\alpha\in\mathbb{N}_0^d, |\alpha|\le N} c_\alpha D^\alpha f.$$

  (This definition has a sense for $\mathcal{C}^N$ functions as well.)

Theorem 25 (properties of the Fourier transform on $\mathscr{S}'$).

(a) *The Fourier transform is a linear bijection of $\mathscr{S}'$ onto $\mathscr{S}'$, for any $\Lambda \in \mathscr{S}'$ we have*

$$\widehat{\widehat{\Lambda}} = \check{\Lambda}, \quad \widehat{\widehat{\widehat{\widehat{\Lambda}}}} = \Lambda.$$

(b) *Let (Λ_n) be sequence in $\mathscr{S}'$ and $\Lambda \in \mathscr{S}'$. Then $\Lambda_n \to \Lambda$ in $\mathscr{S}'$ if and only if $\widehat{\Lambda_n} \to \widehat{\Lambda}$ in $\mathscr{S}'$.*

(c) *If $f \in L^1(\mathbb{R}^d)$, then $\widehat{\Lambda_f} = \Lambda_{\widehat{f}}$.*

(d) *If $f \in L^2(\mathbb{R}^d)$, then $\widehat{\Lambda_f} = \Lambda_{\mathcal{P}(f)}$, where $\mathcal{P}$ is the mapping from the Plancherel theorem.*

(e) *If $\Lambda \in \mathscr{S}'$ and P is a polynomial on $\mathbb{R}^d$, then*

$$\widehat{P(D)\Lambda} = \check{P} \cdot \widehat{\Lambda}, \quad \widehat{P \cdot \Lambda} = \check{P}(D)\widehat{\Lambda}$$

Lemma 26. *Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$.*

(a) *If $\boldsymbol{x}_n \to \boldsymbol{x}$ in $\mathbb{R}^d$, then $\tau_{\boldsymbol{x}_n}\varphi \to \tau_{\boldsymbol{x}}\varphi$ in $\mathscr{S}(\mathbb{R}^d)$.*

(b) *Let $\boldsymbol{e} \in \mathbb{R}^d$. Then $\partial_{\boldsymbol{e}}\varphi \in \mathscr{S}(\mathbb{R}^d)$. Moreover, if we define for each $r \in \mathbb{R} \setminus \{0\}$ the function φ_r by the formula*

$$\varphi_r(\boldsymbol{x}) = \frac{1}{r}(\varphi(\boldsymbol{x} + r\boldsymbol{e}) - \varphi(\boldsymbol{x})), \quad \boldsymbol{x} \in \mathbb{R}^d,$$

i.e., $\varphi_r = \frac{1}{r}(\tau_{-r\boldsymbol{e}}\varphi - \varphi)$, then $\varphi_r \to \partial_{\boldsymbol{e}}\varphi$ in $\mathscr{S}(\mathbb{R}^d)$ for $r \to 0$.

Proposition 27. *Let $d_1, d_2 \in \mathbb{N}$ and $\varphi \in \mathscr{S}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$.*

(a) *Let $\Lambda \in \mathscr{S}'(\mathbb{R}^{d_1})$. For $\boldsymbol{y} \in \mathbb{R}^{d_2}$ we define $\psi(\boldsymbol{y}) = \Lambda(\boldsymbol{x} \mapsto \varphi(\boldsymbol{x}, \boldsymbol{y}))$. Then $\psi \in \mathscr{S}(\mathbb{R}^{d_2})$ and for each multiindex $\alpha \in \mathbb{N}_0^{d_2}$ we have $D^\alpha\psi(\boldsymbol{y}) = \Lambda(\boldsymbol{x} \mapsto D^{(\boldsymbol{o},\alpha)}\varphi(\boldsymbol{x}, \boldsymbol{y}))$ for $\boldsymbol{y} \in \mathbb{R}^{d_2}$.*

(b) (Fubini Theorem for tempered distributions) *Let $\Lambda_1 \in \mathscr{S}'(\mathbb{R}^{d_1})$ and $\Lambda_2 \in \mathscr{S}'(\mathbb{R}^{d_2})$. Then*

$$\Lambda_2(\boldsymbol{y} \mapsto \Lambda_1(\boldsymbol{x} \mapsto \varphi(\boldsymbol{x}, \boldsymbol{y}))) = \Lambda_1(\boldsymbol{x} \mapsto \Lambda_2(\boldsymbol{y} \mapsto \varphi(\boldsymbol{x}, \boldsymbol{y}))).$$

Definition. Let U be a tempered distribution on $\mathbb{R}^d$ and let $\varphi \in \mathscr{S}(\mathbb{R}^d)$. By the **convolution of φ and U** we mean the function $U * \varphi$ defined by the formula

$$U * \varphi(\boldsymbol{x}) = U(\tau_{\boldsymbol{x}}\check{\varphi}) = U(\boldsymbol{y} \mapsto \varphi(\boldsymbol{x} - \boldsymbol{y})), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

Remark. Ig $\varphi \in \mathscr{D}(\mathbb{R}^d)$, then this definition coincides with the definition of the convolution of a distribution and a test function from Section VII.4.

Theorem 28 (on the convolution of a temepered distribution and a function from the Schwartz space). *Let $U \in \mathscr{S}'$, $\varphi, \psi \in \mathscr{S}$. Then:*

(a) *$U * \varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $D^\alpha(U * \varphi) = (D^\alpha U) * \varphi = U * D^\alpha\varphi$ for each multiindex α.*

(b) *$\Lambda_{U*\varphi}$ is a tempered distribution.*

(c) *If $f \in L^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$, then $\Lambda_f * \varphi = f * \varphi$.*

(d) *$\widehat{\Lambda_{U*\varphi}} = \widehat{\varphi} \cdot \widehat{U}$, $\widehat{\varphi \cdot U} = \Lambda_{\widehat{\varphi} * \widehat{U}}$.*

(e) *$U * (\varphi * \psi) = (U * \varphi) * \psi$.*