VIII.3 Lebesgue-Bochner spaces

Definition. Let $f : \Omega \to X$ be strongly μ-measurable.

- Let $p \in [1, \infty)$. We say that the function f belongs to $L^p(\mu; X)$ (more precisely, to $L^p(\Omega, \Sigma, \mu; X)$) provided the function $\omega \mapsto \|f(\omega)\|^p$ is integrable. For such a function we set

$$\|f\|_p = \left(\int_\Omega \|f(\omega)\|^p \, \mathrm{d}\mu \right)^{1/p}.$$

- We say that f belongs to $L^\infty(\mu; X)$ (more precisely, to $L^\infty(\Omega, \Sigma, \mu; X)$) $\omega \mapsto \|f(\omega)\|$ is essentially bounded. For such a function we set

$$\|f\|_\infty = \operatorname*{ess\,sup}_{\omega \in \Omega} \|f(\omega)\|.$$

Remarks:

(1) If $p \in [1, \infty)$, then simple integrable functions belong to $L^p(\mu; X)$. If $f = \sum_{j=1}^k x_j \chi_{E_j}$ where $E_1, \dots, E_k \in \Sigma$ are pairwise disjoint and $x_1, \dots, x_k \in X$, then

$$\|f\|_p = \left(\sum_{j=1}^k \|x_j\|^p \mu(E_j) \right)^{1/p}.$$

(2) Simple measurable functions belong $L^\infty(\mu; X)$. If f is of the above form, then

$$\|f\|_\infty = \max\{\|x_j\| \, ; j \in \{1, \dots, k\} \; \& \; \mu(E_j) > 0\}.$$

(3) If $p \in [1, \infty]$, $h \in L^p(\mu)$ and $x \in X$, then the function $f : \Omega \to X$ defined by the formula $f(\omega) = h(\omega) \cdot x$ belongs tp $L^p(\mu; X)$ and one has $\|f\|_p = \|h\|_p \cdot \|x\|$. We denote $f = h \cdot x$.

Theorem 14.

(a) *Let $p \in [1, \infty]$. After identifying the pairs of functions which are almost everywhere equal, the space $(L^p(\mu; X), \|\cdot\|_p)$ is a Banach space.*

(b) *The space $L^1(\mu; X)$ is formed exactly by the (equivalence classes of) Bochner integrable functions.*

(c) *If X is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, the space $L^2(\mu; X)$ is a Hilbert space as well, the inner product is defined by*

$$\langle f, g \rangle = \int_\Omega \langle f(\omega), g(\omega) \rangle \, \mathrm{d}\mu(\omega), \quad f, g \in L^2(\mu; X).$$

(d) *If μ is finite, then*

$$L^\infty(\mu; X) \subset L^q(\mu; X) \subset L^p(\mu; X) \subset L^1(\mu; X).$$

whenever $1 \le p < q \le \infty$.

Theorem 15. *Let $p \in [1, \infty)$.*

(a) *Simple integrable functions form a dense subspace of $L^p(\mu; X)$.*

(b) *If both spaces $L^p(\mu)$ and X are separable, then $L^p(\mu; X)$ is separable as well.*

Examples 16.

(1) *Let $G \subset \mathbb{R}^n$ be a Lebesgue measurable set of strictly positive measure and let $p \in [1,\infty]$. By $L^p(G;X)$ we denote the space $L^p(\mu;X)$, where μ is the restriction of the n-dimensional Lebesgue measure to G. If $p \in [1,\infty)$ and X is separable, then $L^p(G;X)$ is separable as well.*

(2) *Let μ be the counting measure on $\mathbb{N}$ and let $p \in [1,\infty]$. Then the space $L^p(\mu;X)$ is denoted by $\ell^p(X)$ and can be represented as*

$$\ell^p(X) = \{(x_n) \in X^{\mathbb{N}}; \sum_{n=1}^{\infty} \|x_n\|^p < \infty\} \text{ for } p \in [1,\infty),$$
$$\ell^\infty(X) = \{(x_n) \in X^{\mathbb{N}}; \sup_{n\in\mathbb{N}} \|x_n\| < \infty\}.$$

The respective norm is then defined by the formula

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} \|x_n\|^p\right)^{1/p}, \quad (x_n) \in \ell^p(X), p \in [1,\infty),$$
$$\|(x_n)\|_\infty = \sup_{n\in\mathbb{N}} \|x_n\|, \quad (x_n) \in \ell^\infty(X).$$

If X is separable and $p \in [1,\infty)$, then $\ell^p(X)$ is separable as well.

Remarks on representations of dual spaces. Let $p \in [1,\infty)$ and let $p^* \in (1,\infty]$ be the dual exponent. Then:

(1) The dual to $\ell^p(X)$ is canonically isometric to $\ell^{p^*}(X^*)$. More precisely, if the sequence (φ_n) belongs to $\ell^{p^*}(X^*)$, then the formula

$$(x_n) \mapsto \sum_n \varphi_n(x_n), \qquad (x_n) \in \ell^p(X)$$

defines a continuous linear functional whose norm equals $\|(\varphi_n)\|_{\ell^{p^*}(X^*)}$. Further, any continuous linear functional is of this form.

(2) Assume that X is reflexive and μ is σ-finite. Then the dual to $L^p(\mu;X)$ is canonically isometric to $L^{p^*}(\mu;X^*)$. More precisely, if $g \in L^{p^*}(\mu;X)$, then the formula

$$f \mapsto \int g(\omega)(f(\omega))\,\mathrm{d}\mu, \quad f \in L^p(\mu;X)$$

defines a continuous linear functional whose norm equals $\|g\|_{L^{p^*}(\mu;X^*)}$. Further, any continuous linear functional is of this form.

(3) A proof of (1) is not hard, it is similar to the proof of the representation of the dual to ℓ^p. A proof of (2) is more complicated, it is necessary (among others) to use nontrivial special properties of X. Assertion (2) holds for more general X, but not for every X. The exact formulation of the conditions on X assuring validity of (2) for any σ-finite measure is the following:

$$\forall Y \subset\subset X \text{ separable: } Y^* \text{ is separable.}$$

This condition is equivalent to the **Radon-Nikodým property** of X^*, i.e., to validity of the following version of the Radon-Nikodým theorem:

$$\forall m : \Sigma \to X^* \ \sigma\text{-additive}, m \ll \mu \Rightarrow \exists f \in L^1(\mu, X^*) \forall A \in \Sigma\colon m(A) = (B)\int_A f\,\mathrm{d}\mu.$$

(4) If X is reflexive and $p \in (1,\infty)$, then $L^p(\mu;X)$ is reflexive as well.