V.4 Metrizability of locally convex spaces

Proposition 21. *(1) Let $(X, \mathcal{T})$ be a metrizable LCS. Then the topology $\mathcal{T}$ is generated by a sequence of seminorms (p_n) satisfying*

$$p_1 \le p_2 \le p_3 \le \dots$$

(2) Let X be a vector space and let (p_n) be a sequence of seminorms on X satisfying conditions:

- $p_1 \le p_2 \le p_3 \le;$
- $\forall x \in X \setminus \{\boldsymbol{o}\}\ \exists n \colon p_n(x) > 0.$

Then

$$\rho(x, y) = \sum_{n=1}^{\infty} \tfrac{1}{2^n} \min\{1, p_n(x - y)\}, \quad x, y \in X$$

is a translation invariant metric on X which generates the locally convex topology on X generated by the sequence of seminorms (p_n). Moreover, given a sequence (x_k) in X we have

(a) $\rho(x_k, x) \to 0 \Leftrightarrow \forall n \in \mathbb{N} \colon p_n(x_k - x) \xrightarrow{k} 0;$

(b) *the sequence (x_k) is Cauchy in ρ if and only of it is Cauchy in each of the seminorms p_n.*

Theorem 22 (on metrizability of LCS). *Let $(X, \mathcal{T})$ be a HLCS. The following assertions are equivalent:*

(i) *X is metrizable (i.e., the topology $\mathcal{T}$ is generated by a metric on X).*

(ii) *There exists a translation invariant metric on X generating the topology $\mathcal{T}$.*

(iii) *There exists a countable base of neighborhoods of $\boldsymbol{o}$ in $(X, \mathcal{T})$.*

(iv) *The topology $\mathcal{T}$ is generated by a countable family of seminorms.*

Theorem 23 (a characterization of normable LCS). *Let $(X, \mathcal{T})$ be a HLCS. Then X is normable (i.e., $\mathcal{T}$ is generated by a norm) if and only if X admits a bounded neighborhood of $\boldsymbol{o}$.*

Remark: For general TVS the following statements from this section are valid:

- Equivalence of conditions (i)–(iii) from Theorem 22. The proof is substantially more difficult.
- A variant of Theorem 23 – the existence of a bounded neighborhood of $\boldsymbol{o}$ should be replaced by the existence of a bounded convex neighborhood of zero.