V.5 Fréchet spaces

Definition. Let $(X, \mathcal{T})$ be a LCS. The space X is said to be an **Fréchet-space** if $\mathcal{T}$ is generated by a complete translation invariant metric.

Examples 24.

(1) *Any Banach space is a Fréchet space as well.*
(2) *The spaces $\mathbb{F}^{\mathbb{N}}$, $\mathcal{C}(\mathbb{R}, \mathbb{F})$, $H(\Omega)$ mentioned in Examples 1 are Fréchet spaces.*

Proposition 25. *Let $(X, \mathcal{T})$ be a Fréchet-space. Then any translation invariant metric generating the topology $\mathcal{T}$ is complete.*

Proposition 26. *Let X be a Fréchet-space. Then a set $A \subset X$ is compact if and only if it is totally bounded and closed.*

Proposition 27. *Let X be a LCS and let $A \subset X$ be totally bounded. Then* $\operatorname{aco} A$ *is totally bounded as well.*

Corollary 28. *Let X be a Fréchet space and let $A \subset X$ be a compact subset. Then $\overline{\operatorname{aco} A}$ is compact as well.*

Theorem 29 (Banach-Steinhaus). *Let X be a Fréchet space and let Y be a LCS. Let (T_n) be a sequence of continuous linear mappings $T_n : X \to Y$. Suppose that the limit $\lim\limits_{n \to \infty} T_n x$ exists in Y for each $x \in X$. Then the mapping $T : X \to Y$ defined by the formula $Tx = \lim\limits_{n \to \infty} T_n x$, $x \in X$, is continuous.*

Theorem 30 (open mapping theorem). *Let X and Y be Fréchet spaces and let $T : X \to Y$ be a continuous linear mapping of X onto Y. Then T is an open mapping. In particular, if T is moreover one-to-one, T^{-1} is continuous, i.e., T is an isomorphism of X onto Y.*

Remark: The situation for general TVS is the following:

- A TVS whose topology is generated by a complete translation invariant metric is called F-**space**. Spaces $L^p(\mu)$ for $p \in (0, 1)$ mentioned in Example 1(5) are F-spaces which are not locally convex.
- Propositions 25 and 26 hold for F-spaces as well (no change is needed).
- Proposition 27 and Corollary 28 fail for spaces which are not locally convex.
- Theorem 29 also holds assuming that X is an F-space and Y is a TVS. The proof is similar, but uses a more advanced notion of equicontinuity.
- Theorem 30 holds for F-spaces as well.