VI.3 Polars and their applications

Definition. Let X be a LCS. Let $A \subset X$ and $B \subset X^*$ be nonempty sets. We define

$$\begin{aligned} A^{\triangleright} &= \{f \in X^*; \forall x \in A : \operatorname{Re} f(x) \le 1\}, & B_{\triangleright} &= \{x \in X; \forall f \in B : \operatorname{Re} f(x) \le 1\}, \\ A^{\circ} &= \{f \in X^*; \forall x \in A : |f(x)| \le 1\}, & B_{\circ} &= \{x \in X; \forall f \in B : |f(x)| \le 1\}, \\ A^{\perp} &= \{f \in X^*; \forall x \in A : f(x) = 0\}, & B_{\perp} &= \{x \in X; \forall f \in B : f(x) = 0\}. \end{aligned}$$

The sets $A^{\triangleright}$ and $B_{\triangleright}$ are called **polars** of the sets A and B, the sets A° and $B_{\circ}$ are called **absolute polars** and the sets $A^{\perp}$ and $B_{\perp}$ are called **anihilators**.

Remarks:

(1) The terminology and notaion is not unified in the literature. Sometimes 'the polar' means 'the absolute polar', our polar is sometimes denoted by A°, $B_{\circ}$.

(2) If X is a Hilbert space and $A \subset X$, the symbol $A^{\perp}$ may have two different meanings – it may denote the above-defined anihilator or the orthogonal complement. It should be distinguished according to the context. However, these two possibilities are interrelated as explained in Section III.1. Recall that in this case, given $x \in X$, the formula

$$f_x(y) = \langle y, x \rangle, \quad y \in X$$

defines a continuous linear functional on X and, moreover, $x \mapsto f_x$ is a (conjugate linear) isometry of X onto X^* (see Theorem II.15). Then

$$\text{the anihilator of } A = \{f_x; x \in \text{ the orthogonal complement of } A\}.$$

(3) If X is Hausdorff and if we equip X^* by the weak* topology $\sigma(X^*, X)$, then $(X^*, w^*)^* = \varkappa(X)$, and hence for any $B \subset X^*$ we have $B^{\triangleright} = \varkappa(B_{\triangleright})$, where $B_{\triangleright}$ is the (downward) polar by the previous definition and $B^{\triangleright}$ is the polar with respect to the space (X^*, w^*) and its dual $\varkappa(X)$. Similarly for absolute polars and anihilators.

Example 10. *Let X be a normed linear space. Then*

(a) $(B_X)^{\triangleright} = (B_X)^{\circ} = B_{X^*}$,

(b) $(B_{X^*})_{\triangleright} = (B_{X^*})_{\circ} = B_X$.

Proposition 11 (polar calkulus). *Let X be a LCS and let $A \subset X$ be a nonempty set.*

(a) *The set $A^{\triangleright}$ is convex and contains the zero functional, A° is absolutely convex and $A^{\perp}$ is a subspace of X^*. All the three sets are moreover weak* closed.*

(b) $A^{\perp} \subset A^{\circ} \subset A^{\triangleright}$.

(c) *If A is balanced, then $A^{\triangleright} = A^{\circ}$. If $A \subset\subset X$, then $A^{\triangleright} = A^{\circ} = A^{\perp}$.*

(d) $\{\boldsymbol{o}\}^{\triangleright} = \{\boldsymbol{o}\}^{\circ} = \{\boldsymbol{o}\}^{\perp} = X^*$, $X^{\triangleright} = X^{\circ} = X^{\perp} = \{\boldsymbol{o}\}$.

(e) $(cA)^{\triangleright} = \frac{1}{c} A^{\triangleright}$ *and* $(cA)^{\circ} = \frac{1}{c} A^{\circ}$ *whenever* $c > 0$.

(f) *Let $(A_i)_{i \in I}$ be a nonempty family of nonempty subsets of X. Then $\left(\bigcup_{i \in I} A_i\right)^{\circ} = \bigcap_{i \in I} A_i^{\circ}$. The analogous formulas hold for polars and anihilators.*

Remark: Analogous statements hold for $B \subset X^*$ and for the sets $B_\triangleright$, $B_\circ$, $B_\perp$. There are just two differences: The sets $B_\triangleright$, $B_\circ$ and $B_\perp$ are weakly closed and for the validity of the second statement in (d) one needs to assume that X is Hausdorff.

Theorem 12 (bipolar theorem). *Let X be a LCS and let $A \subset X$ and $B \subset X^*$ be nonempty sets. Then*

$$(A^\triangleright)_\triangleright = \overline{\mathrm{co}}(A \cup \{\boldsymbol{o}\}) \; (= \overline{\mathrm{co}}^{\sigma(X,X^*)}(A \cup \{\boldsymbol{o}\})), \quad (B_\triangleright)^\triangleright = \overline{\mathrm{co}}^{\sigma(X^*,X)}(B \cup \{\boldsymbol{o}\}),$$
$$(A^\circ)_\circ = \overline{\mathrm{aco}} A \; (= \overline{\mathrm{aco}}^{\sigma(X,X^*)} A), \qquad (B_\circ)^\circ = \overline{\mathrm{aco}}^{\sigma(X^*,X)} B,$$
$$(A^\perp)_\perp = \overline{\mathrm{span}} A \; (= \overline{\mathrm{span}}^{\sigma(X,X^*)} A), \qquad (B_\perp)^\perp = \overline{\mathrm{span}}^{\sigma(X^*,X)} B.$$

Corollary 13. *Let X and Y be normed linear spaces and let $T \in L(X,Y)$. Then $(\ker T)^\perp = \overline{T'(Y^*)}^{w^*}$.*

Theorem 14 (Goldstine). *Let X be a normed linear space and let $\varkappa : X \to X^{**}$ be the canonical embedding. Then*

$$B_{X^{**}} = \overline{\varkappa(B_X)}^{\sigma(X^{**},X^*)}.$$

Theorem 15 (Banach-Alaoglu). *Let X be a LCS and let $U \subset X$ be a neighborhood of $\boldsymbol{o}$. Then:*

(a) *U° is a weak* compact subset of X^* (i.e., it is compact in the topology $\sigma(X^*,X)$).*
(b) *If X is moreover separable, U° is metrizable in the topology $\sigma(X^*,X)$.*

Corollary 16 (Banach-Alaoglu for normed spaces). *Let X be a normed linear space. Then (B_{X^*}, w^*) is compact. If X is separable, (B_{X^*}, w^*) is moreover metrizable.*

Corollary 17 (reflexivity and weak compactness). *Let X be a Banach space. Then X is reflexive if and only if B_X is weakly compact. If X is reflexive and separable, (B_X, w) is moreover metrizable.*

Remark: Using Corollaries 16 and 17 we get easy proof of Theorems II.33 and II.34:

- Let X be a separable normed linear space. Then each bounded sequence in X^* admits a weak* convergent subsequence.
- Let X be a reflexive Banach space. Then each bounded sequence in X admits a weakly convergent subsequence.

Corollary 18. *Let X be a reflexive Banach space and let $f : X \to \mathbb{R}$ be a function with the following properties:*

(i) *f is* **weakly sequentially lower semicontinuous**, *i.e.,*

$$\forall x \in X \, \forall (x_n) \text{ a sequence in } X\colon x_n \xrightarrow{w} x \implies f(x) \le \liminf f(x_n).$$

(ii) $\lim\limits_{\|x\| \to \infty} f(x) = +\infty$.

Then f attains its minimum at some point of X.

Condition (i) *is satisfied for example if all level sets $\{x \in X; f(x) \le c\}$, $c \in \mathbb{R}$, are closed and convex.*