A.1 Topological spaces and basic topological notions

Definition. A **topological space** is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a family of subsets of X satisfying the following properties:

(a) $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$.
(b) If $\mathcal{A} \subset \mathcal{T}$ is any subfamily, then $\bigcup \mathcal{A} \in \mathcal{T}$.
(c) For any two sets $U, V \in \mathcal{T}$ we have $U \cap V \in \mathcal{T}$.

A family $\mathcal{T}$ with these properties is called **a topology** on X. Instead of $(X, \mathcal{T})$ we often write just X (if we know which topology is considered).

Definition. Let $(X, \mathcal{T})$ be a topological space.

- A set $A \subset X$ is said to be **open in** $(X, \mathcal{T})$ (or $\mathcal{T}$**-open**, or just **open**), if $A \in \mathcal{T}$.
- Let $A \subset X$ and $x \in A$. The point x is said to be **an interior point** of the set A if there is an open set B such that $x \in B \subset A$.
- **The interior** of a set $A \subset X$ is the set of all its interior points. The interior of A is denoted by $\operatorname{Int} A$ or, more precisely, by $\operatorname{Int}_{\mathcal{T}} A$.
- A set $A \subset X$ is said to be **a neighborhood of the point** $x \in X$ if x is an interior point of A.
- Let $A \subset X$ and $x \in X$. The point x is said to be **a boundary point** of the set A if for each neighborhood U of x we have $U \cap A \neq \emptyset$ and simultaneously $U \cap (X \setminus A) \neq \emptyset$.
- **The boundary** of a set $A \subset X$ is the set of all its boundary points. The boundary of A is denoted by ∂A or, more precisely, by $\partial_{\mathcal{T}} A$. (Sometimes the boundary of A is denoted by $H(A)$ or $\operatorname{bd} A$.)
- A set $A \subset X$ is said to be **closed**, if it contains all its boundary points, i.e. if $\partial A \subset A$.
- **The closure** of a set $A \subset X$ is the set $A \cup \partial A$. The closure of A is denoted by $\overline{A}$ or, more precisely, by $\overline{A}^{\mathcal{T}}$. (Sometimes the closure of A is denoted by $\operatorname{cl} A$ or $\operatorname{cl}_{\mathcal{T}} A$ or $\mathcal{T}\text{-}\operatorname{cl} A$.)

Proposition 1. *Let $(X, \mathcal{T})$ be a topological space and $A \subset X$.*

(i) *The interior of A is the largest open set contained in A.*
(ii) *The set A is closed if and only if $X \setminus A$ is open.*
(iii) *The closure of A is the smallest closed set containing A.*
(iv) *Let $x \in X$. Then $x \in \overline{A}$ if and only if for each neighborhood U of x we have $U \cap A \neq \emptyset$.*

Proposition 2 (properties of closed sets). *Let $(X,\mathcal{T})$ be a topological space.*

(a) *$\emptyset$ and X are closed sets.*
(b) *If $\mathcal{A}$ is any family of closed subsets of X, then $\bigcap \mathcal{A}$ is closed as well.*
(c) *For any two closed sets $C, D \subset X$ the set $C \cup D$ is closed.*

Definition. Let $(X,\mathcal{T})$ be a topological space and $\mathcal{B} \subset \mathcal{T}$.

- The family $\mathcal{B}$ is said to be **a base** (or **basis**) **of the topology** $\mathcal{T}$ if for any $U \in \mathcal{T}$ and any $x \in U$ there exists $G \in \mathcal{B}$ such that $x \in G \subset U$.
- The family $\mathcal{B}$ is said to be **a subbase** (or **subbasis**) **of the topology** $\mathcal{T}$ if for any $U \in \mathcal{T}$ and any $x \in U$ there exist $G_1, \dots, G_k \in \mathcal{B}$ such that $x \in G_1 \cap \dots \cap G_k \subset U$.

Remark. *Let $(X,\mathcal{T})$ be a topological space and $\mathcal{B} \subset \mathcal{T}$.*

- *$\mathcal{B}$ is a base of $\mathcal{T}$ if and only if for each $U \in \mathcal{T}$ there is $\mathcal{A} \subset \mathcal{B}$ with $\bigcup \mathcal{A} = U$.*
- *$\mathcal{B}$ is a subbase of $\mathcal{T}$ if and only if the family of all the sets which can be expressed as the intersection of finitely many elements of $\mathcal{B}$ forms a base of $\mathcal{T}$.*

Proposition 3. *Let X be a set and let $\mathcal{B}$ be a family of subsets of X.*

(i) *The family $\mathcal{B}$ is a base of some topology on X if and only if the following two conditions are fulfilled:*
  - *$\bigcup \mathcal{B} = X$;*
  - *For any $U, V \in \mathcal{B}$ and any $x \in U \cap V$ there exists $W \in \mathcal{B}$ with $x \in W \subset U \cap V$.*

(ii) *The family $\mathcal{B}$ is a subbase of some topology on X if and only if $\bigcup \mathcal{B} = X$.*

Definition. Let $(X,\mathcal{T})$ be a topological space, $a \in X$ and $\mathcal{U}$ be a family of subsets of X. The family $\mathcal{U}$ is said to be **a base of neighborhoods of the point** a if the following two conditions hold:

- Each $U \in \mathcal{U}$ is a neighborhood of a.
- For any neighborhood V of a there is $U \in \mathcal{U}$ with $U \subset V$.

Proposition 4. *Let X be a set and, for each $x \in X$, let $\mathcal{U}_x$ be a family of subsets of X. Then there is a topology $\mathcal{T}$ on X such that for each $x \in X$ the family $\mathcal{U}_x$ is a base of neighborhoods of x, if and only if the following conditions are fulfilled:*

(a) *$x \in U$ whenever $x \in X$ and $U \in \mathcal{U}_x$.*
(b) *If $x \in X$ and $U, V \in \mathcal{U}_x$ then there is $W \in \mathcal{U}_x$ such that $W \subset U \cap V$.*

(c) *For any $x \in X$ and any $U \in \mathcal{U}_x$ there is $V \subset X$ such that $x \in V \subset U$ and, moreover,*

$$\forall y \in V \, \exists W \in \mathcal{U}_y : W \subset V.$$

The topology $\mathcal{T}$ is then uniquely determined and

$$\mathcal{T} = \{U \subset X; \forall x \in U \, \exists V \in \mathcal{U}_x : V \subset U\}.$$

Example. *Let (X, ρ) be a metric space.*

- *For $x \in X$ and $r > 0$ we set $U(x, r) = \{y \in X; \rho(x, y) < r\}$. Then*

$$\mathcal{T} = \{U \subset X; \forall x \in U \, \exists r > 0 : U(x, r) \subset U\}$$

is a topology on X. It is **the topology generated by the metric** *ρ.*

- *Let $x \in X$. Any of the following families is a base of neighborhoods of x:*

$$\{U(x, r); r > 0\}; \qquad \{U(x, \tfrac{1}{n}); n \in \mathbb{N}\}; \qquad \{\overline{U(x, \tfrac{1}{n})}; n \in \mathbb{N}\}.$$

A.2 Continuous mappings

Definition. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and let $f : X \to Y$ be a mapping.

(1) The mapping f is said to be **continuous at** $x \in X$ if for each neighborhood V of $f(x)$ in $(Y, \mathcal{U})$ there exists a neighborhood U of x in $(X, \mathcal{T})$ such that $f(U) \subset V$.

(2) The mapping f is said to be **continuous on** X if it is continuous at each $x \in X$.

Proposition 5 (characterizations of continuity). *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and let $f : X \to Y$ be a mapping. The following assertions are equivalent:*

(i) *f is continuous on X.*

(ii) *For any open set $U \subset Y$ the set $f^{-1}(U)$ is open in X.*

(iii) *For any closed set $F \subset Y$ the set $f^{-1}(F)$ is closed in X.*

(iv) *For any set $A \subset X$ we have $f(\overline{A}) \subset \overline{f(A)}$.*

A.3 Separation axioms

Definition. Let $(X, \mathcal{T})$ be a topological space. The space X is said to be

- T_0, if for any two distinct points $a, b \in X$ there exists $U \in \mathcal{T}$ containing exactly one of the points a, b;
- T_1, if for any two distinct points $a, b \in X$ there exists $U \in \mathcal{T}$ such that $a \in U$ and $b \notin U$;
- T_2 (or **Hausdorff**), if for any two distinct points $a, b \in X$ there exist $U, V \in \mathcal{T}$ such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$;
- **regular**, if for any $a \in X$ and any closed set $B \subset X$ with $a \notin B$ there exist $U, V \in \mathcal{T}$ such that $a \in U$, $B \subset V$ and $U \cap V = \emptyset$;
- T_3, if it is T_1 and regular;
- **completely regular**, if for any $a \in X$ and any closed set $B \subset X$ with $a \notin B$ there exists a continuous function $f : X \to \mathbb{R}$ such that $f(a) = 1$ and $f|_B = 0$;
- $T_{3\frac{1}{2}}$ (or **Tychonoff**), if it is T_1 and completely regular;
- **normal**, if for any two disjoint closed sets $A, B \subset X$ there exist $U, V \in \mathcal{T}$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$;
- T_4, if it is T_1 and normal.

Remark.

- *Trivially* $T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$.
- $T_4 \Rightarrow T_{3\frac{1}{2}}$ *holds as well, but it is not trivial, it is a consequence of the Urysohn lemma.*
- *Any metric space is* T_4.

Proposition 6 (Urysohn lemma). *Let X be a normal topological space and $A, B \subset X$ two disjoint closed sets. Then there exists a continuous function $f : X \to [0, 1]$ such that $f|_A = 0$ and $f|_B = 1$.*

A.4 Subspaces, products and quotients

Definition. Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$. Then $\mathcal{T}_Y = \{U \cap Y; U \in \mathcal{T}\}$ is a topology on Y and the space $(Y, \mathcal{T}_Y)$ is then **a topological subspace** of the space $(X, \mathcal{T})$.

Remark. *Any subspace of a T_0, T_1, T_2, regular, T_3, completely regular or $T_{3\frac{1}{2}}$ space enjoys the same property. (This is obvious.) A subspace of a T_4 space need not be T_4. (This is not obvious.)*

Definition. Let $(X_1, \mathcal{T}_1)$, ... ,$(X_k, \mathcal{T}_k)$ be nonempty topological spaces. By their **cartesian product** we mean the set $X_1 \times \cdots \times X_k$ equipped with the topology, whose base is

$$\{U_1 \times \cdots \times U_k; U_1 \in \mathcal{T}_1, \dots U_k \in \mathcal{T}_k\}.$$

Definition. Let $(X_\alpha, \mathcal{T}_\alpha)$, $\alpha \in A$, be any nonempty family of nonempty topological spaces. By their **cartesian product** we mean the set $\prod_{\alpha \in A} X_\alpha$ equipped with the topology, whose base is

$$\Big\{\{f \in \prod_{\alpha \in A} X_\alpha; f(\alpha_1) \in U_1, \dots, f(\alpha_k) \in U_k\};$$
$$U_1 \in \mathcal{T}_{\alpha_1}, \dots, U_k \in \mathcal{T}_{\alpha_k}, \alpha_1, \dots, \alpha_k \in A, k \in \mathbb{N}\Big\}$$

Proposition 7. *Let $(X_\alpha, \mathcal{T}_\alpha)$, $\alpha \in A$, be any nonempty family of nonempty topological spaces and let $\prod_{\alpha \in A} X_\alpha$ be their cartesian product. Let $(Y, \mathcal{U})$ be a topological space and $f : Y \to \prod_{\alpha \in A} X_\alpha$ a mapping. The mapping f is continuous on Y if and only if for each $\alpha \in A$ the mapping $y \mapsto f(y)(\alpha)$ is a continuous mapping of Y to X_α.*

Definition. Let $(X, \mathcal{T})$ be a topological space, Y a set and $f : X \to Y$ an onto mapping. **The quotient topology** on Y induced by the mapping f is the topology

$$\mathcal{T}_Y = \{U \subset Y; f^{-1}(U) \in \mathcal{T}\}.$$

Definition. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and $f : X \to Y$ an onto mapping. We say that f is **a quotient mapping** if $\mathcal{U}$ is the quotient topology induced by the mapping f.

Proposition 8. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and $f : X \to Y$ a continuous onto mapping. If f is open (i.e., $f(U)$ is open in Y for each open $U \subset X$) or closed (i.e., $f(F)$ is closed in Y for each closed $F \subset X$), then f is a quotient mapping.*

Proposition 9. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and $f : X \to Y$ a quotient mapping. Let $(Z, \mathcal{W})$ be a topological space and let $g : Y \to Z$ be a mapping. Then g is continous if and only if $g \circ f$ is continuous.*

A.5 Compact spaces

Definition. A topological space $(X, \mathcal{T})$ is said to be **compact**, if for any family $\mathcal{U}$ of open sets covering X (i.e. satisfying $\bigcup \mathcal{U} = X$) there exists a finite subfamily $\mathcal{W} \subset \mathcal{U}$ covering X (i.e. such that $\bigcup \mathcal{W} = X$.)

Proposition 10. *Let X be a compact topological space and $Y \subset X$ its topological subspace.*

- *If Y is closed in X, then Y is compact.*
- *If X is Hausdorff and Y is compact, then Y is closed in X.*

Proposition 11. *Let X be a compact topological space, Y a topological space and $f : X \to Y$ a continous onto mapping. Then:*

(i) *Y is compact.*
(ii) *If Y is Hausdorff, then f is a closed mapping (and hence a quotient mapping).*
(iii) *If Y is Hausdorff and f is one-to-one, then f is a homeomorphism (i.e., f^{-1} is continuous as well).*

Proposition 12. *Any Hausdorff compact topological space is T_4, and hence also $T_{3\frac{1}{2}}$.*

Theorem 13 (Tychonoff theorem). *The cartesian product of any family of Hausdorff compact topological spaces is compact. In particular, the spaces $[-1, 1]^\Gamma$, $[0, 1]^\Gamma$, $\{0, 1\}^\Gamma$ and $\{z \in \mathbb{C}; |z| \leq 1\}^\Gamma$ are compact for any set Γ.*

A.6 Convergence of sequences and nets

Definition. Let X be a topological space, (x_n) a sequence of elements of X and $x \in X$. We say that the sequence (x_n) **converges to x in the space X**, if for any neighborhood U of x there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ we have $x_n \in U$. The point x is then called a **limit of the sequence** (x_n), we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

Remark: If X is Hausdorff, then each sequence has at most one limit.

Proposition 14. *Let X be a metric space. Then:*

(1) *Let $A \subset X$. Then*

$$\overline{A} = \{x \in X; \exists (x_n) \text{ a sequence in } A : x_n \to x\}$$

(2) *Let $A \subset X$. Then A is closed if and only if any $x \in X$, which is the limit of a sequence in A, belongs to A.*

(3) *Let Y be a topological space, $f : X \to Y$ a mapping and $x \in X$. The mapping f is continuous at x, if and only if*

$$\forall (x_n) \text{ sequence in } X : x_n \to x \Rightarrow f_n(x) \to f(x).$$

Definition. Let $(\Gamma, \preceq)$ be a partially ordered set. We say that it is **directed** (more precisely **up-directed**), if for any pair $\gamma_1, \gamma_2 \in \Gamma$ thre exists $\gamma \in \Gamma$ such that $\gamma_1 \preceq \gamma$ a $\gamma_2 \preceq \gamma$.

Examples of directed sets:.

- $\Gamma =$ *the set of all finite subsets of* $\mathbb{N}$, $A \preceq B \equiv^{\text{df}} A \subset B$.
- $\Gamma =$ *the set of all neighborhoods of x in a topological space X*, $U \preceq V \equiv^{\text{df}} U \supset V$.

Definition. Let X be a topological space and let $(\Gamma, \preceq)$ be a directed set.

- By a **net indexed by** Γ we mean any mapping $\alpha : \Gamma \to X$.
- We say that a net $\alpha : \Gamma \to X$ **converges to** $x \in X$ if

$$\forall U \text{ neighborhood of } x \, \exists \gamma_0 \in \Gamma \, \forall \gamma \in \Gamma, \gamma \succeq \gamma_0 : \alpha(\gamma) \in U.$$

The point x is called a **limit of the net** α, we write $\lim_{\gamma\in\Gamma} \alpha(\gamma) = x$ or $\alpha(\gamma) \xrightarrow{\gamma\in\Gamma} x$.

Remark: If X is Hausdorff, then each net in X has at most one limit.

Proposition 15. *Let X be a topological space. Then:*

(1) *Let $A \subset X$. Then*

$$\overline{A} = \{x \in X; \exists \text{ a net } \alpha : \Gamma \to A : \alpha(\gamma) \xrightarrow{\gamma\in\Gamma} x\}$$

(2) *Let $A \subset X$. Then A is closed if and only if any $x \in X$, which is a limit of a net in A, belongs to A.*

(3) *Let Y be a topological space, $f : X \to Y$ a mapping and $x \in X$. The mapping f is continuous at x, if and only if*

$$\forall \text{ net } \alpha : \Gamma \to X : \alpha(\gamma) \xrightarrow{\gamma\in\Gamma} x \Rightarrow f(\alpha(\gamma)) \xrightarrow{\gamma\in\Gamma} f(x).$$