FUNCTIONAL ANALYSIS 1

WINTER SEMESTER 2023/2024

PROBLEMS TO CHAPTER VIII

Problems to Section VIII.1 – measurability of vector-valued functions

Problem 1. Let Γ be a set, let Σ be the σ-algebra of subsets of $\Gamma \times \Gamma$ generated by all the rectangles (i.e., by the sets of the form $A \times B$, $A, B \subset \Gamma$) and let $\Delta = \{(\gamma, \gamma), \gamma \in \Gamma\}$ be the diagonal of $\Gamma \times \Gamma$. Show that $\Delta \in \Sigma$ if and only if the cardinality of Γ is at most equal to the cardinality of continuum.

`Hint:` *IF PART: The assumption that the cardinality of Γ is at most equal to the cardinality of continuum means that we can assume $\Gamma \subset \mathbb{R}$. Further, there are countable partitions $\mathcal{P}_n = \{A_{n,m}; m \in \mathbb{N}\}$ of $\mathbb{R}$ such that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ and whenever $A_{n,m_n} \in \mathcal{P}_n$ for each n, then the intersection $\bigcap_{n\in\mathbb{N}} A_{n,m_n}$ contains at most one point. ONLY IF PART: Show that for any $M \in \Sigma$ there is a partition $(A_j)_{j\in J}$ of Γ such that the cardinality of J is at most equal to the cardinality of continuum and for each $j \in J$ either $A_j \times A_j \subset M$ or $(A_j \times A_j) \cap M = \emptyset$. To prove this show that the described property is satisfied by rectangles and it is preserved by taking complements and countable unions. The key tool to prove the stability to countable union is the fact that (i) a countable union of sets of cardinality at most continuum has again cardinality at most continuum and (ii) the set of sequences of elements of a set of cardinality at most continuum has again cardinality at most continuum.*

Problem 2. Let X be a Banach space of cardinality strictly larger than continuum, let $\Omega = X \times X$ and let Σ be the σ-algebra of subsets of Ω generated by Borel rectangles (i.e., by the sets $A \times B$, where $A, B \subset X$ are Borel sets). Define two functions $f, g : \Omega \to X$ by

$$f(x, y) = x, \quad g(x, y) = y \qquad \text{for } (x, y) \in \Omega.$$

Show that both f and g are Borel Σ-measurable, but $f - g$ is not Borel Σ-measurable.

`Hint:` *Use Problem 1.*

Problem 3. Let (Ω, Σ, μ) be a complete measure space, let X be a Banach space and let $f : \Omega \to X$ be a mapping. A point $x \in X$ belongs to the **essential range** of f if for any U, neighborhood of x in X, the inverse image $f^{-1}(U)$ is not a set of zero μ-measure.

(1) Show that the essential range of f is separable whenever μ is a finite measure and f is Borel Σ-measurable.
(2) Suppose that f is essentially separably valued (i.e., f has **essentially separable range**). Show that the essential range of f is separable.
(3) Find an example of a function which fails to have essentially separable range but whose essential range is empty (and hence separable).

`Hint:` *(1) If the essential range is nonseparable, it contains an uncountable ε-discrete set D for some $\varepsilon > 0$. Then $f^{-1}(U(d, \varepsilon/2))$, $d \in D$, is an uncountable system of measurable sets of strictly positive measure. (3) Consider, for example, the function $f : [0, 1] \to \ell^2([0, 1])$ defined by $f(t) = e_t$, see Example VIII.6(1).*

Problem 4. Show that the following two assertions are equivalent:

(i) There exists a finite complete measure space (Ω, Σ, μ), a Banach space X and a Borel Σ-measurable function $f : \Omega \to X$ which is not essentially separably valued.

(ii) There is a set Γ and a finite non-zero σ-additive measure defined on the σ-algebra of all the subsets of Γ which is zero on all the singletons (i.e., there exists a **real-valued measurable cardinal**).

Hint: *(ii)⇒(i) Observe that Γ must be uncountable and take $f : \Gamma \to \ell^2(\Gamma)$ with $f(\gamma) = e_\gamma$. (i)⇒(ii) Let R denote the essential range of f. By Problem 3(1) R is separable, hence $\Omega \setminus f^{-1}(R)$ has positive measure. Thus without loss of generality $R = \emptyset$. In other words, any point $x \in X$ has a neighborhood whose preimage has zero measure. Using the fact that any metric space has a σ-disjoint base of open sets one can find disjoint families $\mathcal{U}_n$ of open sets with zero measure of preimages such that $\bigcup_n \bigcup \mathcal{U}_n = X$. There is n with $\mu(f^{-1}(\bigcup \mathcal{U}_n)) > 0$. Set $\Gamma = \mathcal{U}_n$ and $\nu(A) = \mu(f^{-1}(\bigcup A))$ for $A \subset \Gamma$.*

Problem 5. Let (Ω, Σ) be a measurable space, let X be a Banach space and let (f_n) be a sequence of strongly Σ-measurable functions $f_n : \Omega \to X$ such that for each $\omega \in \Omega$ the sequence $(f_n(\omega))$ weakly converges to some $f(\omega)$. Show that f is strongly Σ-measurable.

Hint: *Show that f is weakly Σ-measurable and has separable range (using the Mazur theorem). Then use the Pettis theorem.*

Problem 6. Let (Ω, Σ, μ) be a complete measure space, let X be a Banach space and let (f_n) be a sequence of strongly μ-measurable functions $f_n : \Omega \to X$ such that for almost all $\omega \in \Omega$ the sequence $(f_n(\omega))$ weakly converges to some $f(\omega)$. Show that f is strongly μ-measurable.

Hint: *Show that f is weakly μ-measurable and is essentially separably valued (using the Mazur theorem). Then use the Pettis theorem.*

Problem 7. Let (Ω, Σ, μ) be the interval $(0, \infty)$ with the Lebesgue measure, $X = L^p((0, \infty))$ with $p \in [1, \infty)$ and $\psi : (0, \infty) \to \mathbb{F}$ a function.

(1) Show that the function $\phi : t \mapsto \psi(t) \cdot \chi_{(0,t)}$ is μ-measurable if and only if ψ is Lebesgue measurable.

(2) Show that the function $\phi : t \mapsto \psi \cdot \chi_{(0,t)}$ (i.e., $t \mapsto (u \mapsto \psi(u)\chi_{(0,t)}(u))$) is μ-measurable if and only if $\psi|_{(0,T)} \in L^p((0, T))$ for each $T \in (0, \infty)$.

(3) Suppose that ψ has values in $(0, \infty)$. Show that the function $\phi : t \mapsto \chi_{(0,\psi(t))}$ is μ-measurable if and only if ψ is Lebesgue-measurable.

Hint: *(1) ONLY IF PART: $\phi(t) \in X$ for each t. To prove measurability of ψ consider functionals represented by $\chi_{(0,T)}$, $T \in (0, \infty)$. IF PART: Prove the weak measurability. (2) ONLY IF PART: The condition is necessary in order $\phi(t) \in X$ for each t. IF PART: Prove the weak measurability. (3) ONLY IF PART: $\phi(t) \in X$ for each t. To prove measurability of ψ consider functionals represented by $\chi_{(0,T)}$, $T \in (0, \infty)$. IF PART: It is enough to prove weak measurability. It is easy to see that ϕ is weakly measurable for ψ continuous. Further, if $\psi_n \to \psi$ almost everywhere on $(0, \infty)$, then for the respective functions ϕ_n and ϕ one has $\phi_n(t) \to \phi(t)$ weakly for almost all t.*

Problem 8. Let (Ω, Σ, μ) be the interval $(0, \infty)$ with the Lebesgue measure, $X = L^\infty((0,\infty))$ with $p \in (1, \infty)$ and $\psi : (0, \infty) \to \mathbb{F}$ a function.

(1) Show that the function $\phi : t \mapsto \psi(t) \cdot \chi_{(0,t)}$ is strongly μ-measurable if and only if $\psi = 0$ almost everywhere.
(2) Show that the function $\phi : t \mapsto \psi \cdot \chi_{(0,t)}$ (i.e., $t \mapsto (u \mapsto \psi(u)\chi_{(0,t)}(u))$) is strongly μ-measurable if and only if $\psi = 0$ almost everywhere.
(3) Suppose that ψ has values in $(0, \infty)$. Show that the function $\phi : t \mapsto \chi_{(0,\psi(t))}$ is strongly μ-measurable if and only if ψ is Lebesgue-measurable and there is a countable set $C \subset (0, \infty)$ such that $\psi(t) \in C$ for almost all $t \in (0, \infty)$.

`Hint:` *(1) IF PART: If $\psi = 0$ almost everywhere, then $\phi = 0$ almost everywhere. ONLY IF PART: $\|\phi(t) - \phi(u)\| \geq |\psi(u)|$ whenever $t < u$. If $\psi(t) \neq 0$ on a set which does not have measure zero, then for some $n \in \mathbb{N}$ one has $|\psi(t)| > \frac{1}{n}$ on a set which does not have measure zero, hence ϕ is not essentially separably valued. (2) IF PART: If $\psi = 0$ almost everywhere, then $\phi = 0$ almost everywhere. ONLY IF PART: First observe that ψ must be Lebesgue-measurable. Futher $\|\phi(t) - \phi(u)\| = \left\|\psi|_{(t,u)}\right\|$ whenever $t < u$. If $\psi \neq 0$ on a set of positive measure, there is $n \in \mathbb{N}$ such that $A = \{t; |\psi(t)| \geq \frac{1}{n}\}$ has positive measure. Let Z be the union of all the open intervals I such that $I \cap A$ has zero measure. Then $Z \cap A$ has zero measure as well. Then $(0, \infty) \setminus Z$ is a closed set of positive measure. Let B denote the set of all the one-sided isolated points of $(0, \infty) \setminus Z$. Then B is countable, hence $C = (0, \infty) \setminus (Z \cup B)$ is a set of positive measure. Moreover, if $u, t \in C$, $u < t$, then $(u, t) \cap A$ has positive measure. It follows that ϕ is not essentially separably valued. (3) IF PART: Show that ϕ is essentially separably valued and Borel μ-measurable. ONLY IF PART: $\phi(t) \in X$ for each t. To prove measurability of ψ consider functionals represented by $\chi_{(0,T)}$, $T \in (0, \infty)$. Furhter, characteristic functions form a discrete set in X, so if ϕ is essentially separably valued, we find the respective set C.*

Problem 9. Let K be a compact Hausdorff space and let $X = \mathcal{C}(K)$ be the space of continuous functions on K equipped with the sup-norm. Further, let (Ω, Σ) be a measurable space and $f : \Omega \to X$ a mapping.

(1) Suppose that f is weakly Σ-measurable. Show that for each $k \in K$ the mapping $\omega \mapsto f(\omega)(k)$ is Σ-measurable.
(2) Suppose moreover that K is metrizable. Show that the converse holds as well. I.e., f is weakly measurable provided that the mapping $\omega \mapsto f(\omega)(k)$ is Σ-measurable for each $k \in K$.

`Hint:` *(2) By Riesz theorem we need to show that $\omega \mapsto \int_K f(\omega)(k)\, \mathrm{d}\mu(k)$ is measurable for each (signed or complex) Radon measure on K. The set of all the measures with this property contains Dirac measures (by the assumption). Further, it is a linear subspace and it is closed with respect to weak* limits of sequences. Further, the absolute convex hull of Dirac measures is weak* closed in the unit ball $B_{\mathcal{C}(K)^*}$ by the bipolar theorem. Finally, since $\mathcal{C}(K)$ is separable, the dual unit ball is metrizable in the weak* topology, hence any measure from the unit ball is the weak* limit of a sequence from the absolutely convex hull of Dirac measures.*

Problem 10. Let (Ω, Σ, μ) be the interval $[0, 1]$ with the Lebesgue measure, $X = \mathcal{C}([0, 1])$ and $f : [0, 1]^2 \to \mathbb{F}$ be a function. Show that the function

$$\Phi : t \mapsto f(t, \cdot)$$

is a μ-measurable function $\Omega \to X$ if and only if

- $u \mapsto f(t, u)$ is continuous on $[0, 1]$ for each $t \in [0, 1]$;
- $t \mapsto f(t, u)$ is Lebesgue measurable for each $u \in [0, 1]$.

Problem 11. Let (Ω, Σ, μ) be a complete measure space and let X be a reflexive Banach space. Show that every weakly integrable function $f : \Omega \to X$ is Pettis integrable.

Problem 12. Let (Ω, Σ, μ) be the interval $(0,1)$ with the Lebesgue measure and let $X = L^p((0,1))$ where $p \in (1, \infty)$. Let $\psi : (0,1) \to \mathbb{F}$ be a Lebesgue measurable function. Define the function $\phi : (0,1) \to X$ by

$$\phi(t)(u) = \psi(t)\chi_{(0,t)}(u), \quad u \in (0,1), t \in (0,1).$$

(1) Show that ϕ is Pettis-integrable if and only if $\int_0^1 \left(\int_u^1 |\psi|\right)^p \,\mathrm{d}u < \infty$.
(2) Show that ϕ is Bochner integrable if and only if $\int_0^1 t^{1/p} |\psi(t)| \,\mathrm{d}t < \infty$.
(3) Is in this case the Bochner integrability equivalent to the Pettis integrability?
(4) Compute the respective Pettis or Bochner integral.

`Hint:` *(1) By Problem 11 it is enough to show the weak integrability. In the proof one can use the known fact that $fg \in L^1$ for each $g \in L^q$ if and only if $f \in L^p$ (where $\frac{1}{p} + \frac{1}{q} = 1$). (4) Compute it as a Pettis integral.*

Problem 13. Let (Ω, Σ, μ) be the interval $(0,1)$ with the Lebesgue measure and let $X = L^p((0,1))$ where $p \in (1, \infty)$. Let $\psi : (0,1) \to \mathbb{F}$ be a Lebesgue measurable function such that $\psi|_{(0,r)} \in L^p((0,r))$ for each $r \in (0,1)$. Define the function $\phi : (0,1) \to X$ by

$$\phi(t)(u) = \psi(u)\chi_{(0,t)}(u), \quad u \in (0,1), t \in (0,1).$$

(1) Show that ϕ is Pettis-integrable if and only if $\int_0^1 (1-u)^p |\psi(u)|^p \,\mathrm{d}u < \infty$.
(2) Show that ϕ is Bochner integrable if and only if $\int_0^1 \left(\int_0^t |\psi|^p\right)^{1/p} \mathrm{d}t < \infty$.
(3) Is in this case the Bochner integrability equivalent to the Pettis integrability?
(4) Compute the respective Pettis or Bochner integral.

`Hint:` *(1) By Problem 11 it is enough to show the weak integrability. In the proof one can use the known fact that $fg \in L^1$ for each $g \in L^q$ if and only if $f \in L^p$ (where $\frac{1}{p} + \frac{1}{q} = 1$). (4) Compute it as a Pettis integral.*

Problem 14. Let (Ω, Σ, μ) be the interval $(0,1)$ with the Lebesgue measure and let $X = L^p((0,1))$ where $p \in [1, \infty)$. Let $\psi : (0,1) \to (0,1]$ be a Lebesgue measurable function. Define the function $\phi : (0,1) \to X$ by

$$\phi(t)(u) = \chi_{(0,\psi(t))}(u), \quad u \in (0,1), t \in (0,1).$$

Show that ϕ is Bochner integrable and compute the Bochner integral.

Problem 15. Let (Ω, Σ, μ) be the interval $(0, \infty)$ with the Lebesgue measure and let $X = L^p((0, \infty))$ where $p \in (1, \infty)$. Let $\psi : (0, \infty) \to (0, \infty)$ be a Lebesgue measurable function. Define the function $\phi : (0, \infty) \to X$ by

$$\phi(t)(u) = \chi_{(0,\psi(t))}(u), \quad u \in (0, \infty), t \in (0, \infty).$$

(1) Show that ϕ is Pettis-integrable if and only if $\int_0^\infty \mu(\psi^{-1}(u, \infty))^p \, \mathrm{d}u < \infty$.
(2) Show that ϕ is Bochner integrable if and only if $\int_0^\infty |\psi(t)|^{1/p} \, \mathrm{d}t < \infty$.
(3) Is in this case the Bochner integrability equivalent to the Pettis integrability?
(4) Compute the respective Pettis or Bochner integral.

`Hint:` *(1) By Problem 11 it is enough to show the weak integrability. In the proof one can use the known fact that $fg \in L^1$ for each $g \in L^q$ if and only if $f \in L^p$ (where $\frac{1}{p} + \frac{1}{q} = 1$). (4) Compute it as a Pettis integral.*

Problem 16. Let (Ω, Σ, μ) be the interval $(0, 1)$ with the Lebesgue measure and let $X = L^1((0, 1))$. Let $\psi : (0, 1) \to \mathbb{F}$ be a Lebesgue measurable function. Define the function $\phi : (0, 1) \to X$ by

$$\phi(t)(u) = \psi(t)\chi_{(0,t)}(u), \quad u \in (0, 1), t \in (0, 1).$$

(1) Show that ϕ is weakly integrable if and only if $\int_0^1 t \, |\psi(t)| \, \mathrm{d}t < \infty$.
(2) Show that ϕ is Bochner integrable whenever it is weakly integrable.
(3) Compute the respective Bochner integral.

Problem 17. Let (Ω, Σ, μ) be the interval $(0, 1)$ with the Lebesgue measure and let $X = L^1((0, 1))$. Let $\psi : (0, 1) \to \mathbb{F}$ be a Lebesgue measurable function such that $\psi|_{(0,r)} \in L^1((0, r))$ for each $r \in (0, 1)$. Define the function $\phi : (0, 1) \to X$ by

$$\phi(t)(u) = \psi(u)\chi_{(0,t)}(u), \quad u \in (0, 1), t \in (0, 1).$$

(1) Show that ϕ is weakly integrable if and only if $\int_0^1 (1 - u) \, |\psi(u)| \, \mathrm{d}u < \infty$.
(2) Show that ϕ is Bochner integrable whenever it is weakly integrable.
(3) Compute the respective Bochner integral.

Problem 18. Let (Ω, Σ, μ) be the interval $(0, \infty)$ with the Lebesgue measure and let $X = L^1((0, \infty))$. Let $\psi : (0, \infty) \to (0, \infty)$ be a Lebesgue measurable function. Define the function $\phi : (0, \infty) \to X$ by

$$\phi(t)(u) = \chi_{(0,\psi(t))}(u), \quad u \in (0, \infty), t \in (0, \infty).$$

(1) Show that ϕ is weakly integrable if and only if $\int_0^\infty \mu(\psi^{-1}(u, \infty)) \, \mathrm{d}u < \infty$.
(2) Show that ϕ is Bochner integrable if and only if $\psi \in L^1((0, \infty))$.
(3) Show that in this case the Bochner integrability equivalent to weak integrability.
(4) Compute the respective Bochner integral.

Problem 19. Let (Ω, Σ, μ) be the interval $(0, 1)$ with the Lebesgue measure and let $X = L^\infty((0, 1))$. Let $\psi : (0, 1) \to (0, 1)$ be a Lebesgue measurable function which is "essentially countably valued" (see the condition in Problem 8(3)). Define the function $\phi : (0, 1) \to X$ by

$$\phi(t)(u) = \chi_{(0,\psi(t))}(u), \quad u \in (0, 1), t \in (0, 1).$$

Show that ϕ is Bochner integrable and compute the Bochner integral.

Problem 20. Let (Ω, Σ, μ) be the interval $(0, \infty)$ with the Lebesgue measure and let $X = L^\infty((0, \infty))$. Let $\psi : (0, \infty) \to (0, \infty)$ be a Lebesgue measurable function which is "essentially countably valued" (see the condition in Problem 8(3)). Define the function $\phi : (0, \infty) \to X$ by

$$\phi(t)(u) = \chi_{(0,\psi(t))}(u), \quad u \in (0, \infty), t \in (0, \infty).$$

Characterize the functions ψ for which ϕ is Bochner integrable and compute the Bochner integral.

Problem 21. Let (Ω, Σ, μ) be a complete σ-finite measure space, let K be a metrizable compact space and let $f : \Omega \times K \to \mathbb{F}$ be a mapping satisfying

- $t \mapsto f(\omega, t)$ is continuous on K for each $\omega \in \Omega$;
- $\omega \mapsto f(\omega, t)$ is μ-measurable for each $t \in K$.

Let us define a mapping $\Phi : \Omega \to \mathcal{C}(K)$ by

$$\Phi(\omega) = f(\omega, \cdot), \quad \omega \in \Omega.$$

(1) Show that Φ is weakly integrable if and only if $\sup_{t\in K} \int_\Omega |f(\omega, t)| \,\mathrm{d}\mu(\omega) < \infty$.
(2) Suppose that Φ is weakly integrable. Set $g(t) = \int_\Omega f(\omega, t) \,\mathrm{d}\mu(\omega)$ for $t \in K$. Show that g is universally measurable (i.e., measurable for each Radon probability measure on K) and the weak integral of Φ over Ω is the element of $\mathcal{C}(K)^{**} = \mathcal{M}(K)^*$ defined by

$$\nu \mapsto \int_K g \,\mathrm{d}\nu, \quad \nu \in \mathcal{M}(K).$$

(3) Find an example when g is not continuous.
(4) Find an example when g is continuous but Φ is not Pettis integrable.
(5) Show that Φ is Pettis integrable if and only if the mapping $t \mapsto \int_A f(\omega, t) \,\mathrm{d}\mu(\omega)$ is continous on K for each $A \in \Sigma$.
(6) Show that Φ is Bochner integrable if and only if $\int_\Omega \sup_{t\in K} |f(\omega, t)| \,\mathrm{d}\mu(\omega) < \infty$.
(7) Is Bochner integrability equivalent to Pettis integrability in this case?

`Hint:` *(1) ONLY IF PART: If Φ is weakly integrable, then the mapping $\varphi \mapsto \varphi \circ \Phi$ is a continuous linear operator from X^* to $L^1(\mu)$ (cf. the proof of Proposition VIII.11). IF PART: One needs to show that for each $\nu \in \mathcal{M}(K)$ the mapping $h_\nu : \omega \mapsto \int_K f(\omega, t) \,\mathrm{d}\nu(t)$ belongs to $L^1(\mu)$. By the assumption this is satisfied for Dirac measures and hence also for their linear combinations. Further, if $\nu_n \xrightarrow{w^*} \nu$ in $\mathcal{M}(K) = \mathcal{C}(K)^*$, then $h_{\nu_n} \to h_\nu$ pointwise. If $\nu \in M(K)$ is arbitrary, there is a sequence (ν_n) of linear combinations of Dirac measures such that $\nu_n \xrightarrow{w^*} \nu$ and $\|\nu_n\| \le \|\nu\|$ for each n. Finally, the sequence (h_{ν_n}) is bounded in $L^1(\mu)$, hence $h_\nu \in L^1(\mu)$ by the Fatou lemma. (2) Use the Fubini theorem. (5) Use (2) and the definitions. (6) Use Theorem VIII.8.*

Problem 22. Let (Ω, Σ, μ) be a complete finite measure space, let $p \in [1, \infty]$ and let $X = c_0$. Any function $\boldsymbol{f} : \Omega \to X$ can be canonically identified with a sequence (f_n) of scalar-valued functions on Ω.

(1) Show that a function $\boldsymbol{f} : \Omega \to X$ is μ-measurable if and only if f_n is μ-measurable for each $n \in \mathbb{N}$.
(2) Let $\boldsymbol{f} : \Omega \to X$ be μ-measurable. Compute $\|\boldsymbol{f}\|_p$ and using this formula describe the space $L^p(\mu; X)$.
(3) Show that $L^\infty(\mu, X)$ is canonically isometric to a proper subspace of $\left(\bigoplus_{n\in\mathbb{N}} L^\infty(\mu)\right)_{\ell^\infty}$ and describe this subspace.

`Hint:` *(1) Consider the weak measurability.*

Problem 23. Let (Ω, Σ, μ) be a complete finite measure space, let $p \in [1, \infty]$, $q \in [1, \infty)$ and let $X = \ell^q$. Any function $\boldsymbol{f} : \Omega \to X$ can be canonically identified with a sequence (f_n) of scalar-valued functions on Ω.

(1) Show that a function $\boldsymbol{f} : \Omega \to X$ is μ-measurable if and only if f_n is μ-measurable for each $n \in \mathbb{N}$.
(2) Let $\boldsymbol{f} : \Omega \to X$ be μ-measurable. Compute $\|\boldsymbol{f}\|_p$ and using this formula describe the space $L^p(\mu; X)$.
(3) Assuming $p = q$ show that $L^p(\mu; X)$ is canonically isometric to the space $\left(\bigoplus_{n\in\mathbb{N}} L^p(\mu)\right)_{\ell^p}$.

`Hint:` *(1) Consider the weak measurability.*

Problem 24. Let (Ω, Σ, μ) be a complete finite measure space, let $p \in [1, \infty]$ and let $X = \ell^\infty$. Any function $\boldsymbol{f} : \Omega \to X$ can be canonically identified with a sequence (f_n) of scalar-valued functions on Ω.

(1) Show that a function $\boldsymbol{f} : \Omega \to X$ is μ-measurable if and only if it is essentially separably valued and f_n is μ-measurable for each $n \in \mathbb{N}$.
(2) Let $\boldsymbol{f} : \Omega \to X$ be μ-measurable. Compute $\|\boldsymbol{f}\|_p$ and using this formula describe the space $L^p(\mu; X)$.
(3) Show that $L^\infty(\mu, X)$ is canonically isometric to a proper subspace of $\left(\bigoplus_{n\in\mathbb{N}} L^\infty(\mu)\right)_{\ell^\infty}$ and describe this subspace.

`Hint:` *(1) The necessity is clear. To prove the sufficiency observe that the inverse image of any open ball in X by $\boldsymbol{f}$ belongs to Σ and using the assumption that $\boldsymbol{f}$ is essentially separably valued prove the Borel μ-measurability.*

Problem 25. Let (Ω, Σ, μ) be a complete σ-finite measure space, let $p \in [1, \infty]$, let $p^* \in [1, \infty]$ denote the conjugate exponent, let X be a Banach space and let X^* denote its dual.

(1) Let $f \in L^p(\mu; X)$ and $g \in L^{p^*}(\mu; X^*)$. Show that the function
$$h(\omega) = g(\omega)(f(\omega)), \quad \omega \in \Omega,$$
is μ-integrable and, moreover $\|h\|_1 \leq \|g\|_{p^*} \cdot \|f\|_p$.

(2) For $g \in L^{p^*}(\mu; X^*)$ define
$$\Phi_g(f) = \int_\Omega g(\omega)(f(\omega)) \, \mathrm{d}\mu(\omega), \quad f \in L^p(\mu; X).$$
Show that Φ_g is a continuous linear functional on $L^p(\mu; X)$ and $\|\Phi_g\| \leq \|g\|_{p^*}$.

(3) Show that $\|\Phi_g\| = \|g\|_{p^*}$ and deduce that $\Phi : g \mapsto \Phi_g$ is a linear isometry of $L^{p^*}(\mu; X^*)$ into $L^p(\mu, X)^*$.

(4) Show that the mapping Φ is onto $L^p(\mu, X)^*$ provided $p \in [1, \infty)$ and $X = c_0$ or $X = \ell^q$ for some $q \in (1, \infty)$.

(5) Show that the mapping Φ is onto $L^p(\mu, X)^*$ provided $p \in [1, \infty)$ and (Ω, Σ, μ) is the set $\mathbb{N}$ with the counting measure.

(6) Show that the mapping Φ is not onto provided (Ω, Σ, μ) is the interval $[0, 1]$ with the Lebesgue measure and $X = \ell^1$.

`Hint:` *(1) Firstly, one need to show the measurability of h. If g is a simple function, it is easy. To prove the general case use the existence of a sequence of simple integrable functions converging almost everywhere to g. For the estimate use Hölder inequality. (2) follows from (1). (3) Since countably-valued functions are dense in $L^{p^*}(\mu; X^*)$, it is enough to prove the equality for countably-valued g. One inequality follows from (2), it is enough to prove the converse one. Fix $\varepsilon > 0$. By the scalar case there is a non-negative measurable function h with $\|h\|_p = 1$ such that $\int_\Omega h(\omega) \|g(\omega)\| \, \mathrm{d}\mu(\omega) > \|g\|_{p^*} - \varepsilon$. Let $g = \sum_{j=1}^\infty x_j^* \chi_{E_j}$, where $x_j^* \in X^*$ and $E_j \in \Sigma$ with $0 < \mu(E_j) < \infty$ for each $j \in \mathbb{N}$. Fix a sequence of positive numbers $\delta_j > 0$ which are small enough and find $x_j \in X$ with $\|x_j\| = 1$ and $x_j^*(x_j) > \left\|x_j^*\right\| - \delta_j$. Take $f = \sum_{j=1}^\infty h x_j \chi_{E_j}$. Then $f \in L^p(\mu; X)$, $\|f\|_p = 1$ and $\Phi_g(f) > \|g\|_{p^*} - 2\varepsilon$. (4) Use Problems 22 and 23 and the method of proving the representation of duals of c_0 and ℓ^p. (5) Use Example VIII.16(2). (6) Using Problem 23 and the method of proving the representation of dual of ℓ^1 describe the dual of $L^p(\mu; \ell^1)$ and compare it with the space $L^{p^*}(\mu; \ell^\infty)$ described in Problem 24.*