FUNCTIONAL ANALYSIS 1

WINTER SEMESTER 2023/2024

PROBLEMS TO CHAPTER IX

Problem 1. Let X be a Banach space and $K \subset X$ a nonempty weakly compact convex set. Show that $\overline{\operatorname{co}\operatorname{ext} K}^{\|\cdot\|} = K$.

`Hint:` *Combine Krein-Milman and Mazur theorems.*

Problem 2. Let X be a reflexive Banach space. Show that $\overline{\operatorname{co}\operatorname{ext} B_X}^{\|\cdot\|} = B_X$.

`Hint:` *Use Problem 1.*

Problem 3. Let $p \in (1, \infty)$ and let μ be any σ-additive measure such that there exists a measurable set A with $0 < \mu(A) < \infty$. Show the set $\operatorname{ext} B_{L^p(\mu)}$ coincides with the unit sphere.

Problem 4. Let μ be a σ-additive measure which is not constant zero.

(1) Describe the extreme points of $B_{L^\infty(\mu)}$ both in the real and complex cases.
(2) Is $\overline{\operatorname{co}\operatorname{ext} B_{L^\infty(\mu)}}^{\|\cdot\|} = B_{L^\infty(\mu)}$?

`Hint:` *(2) Show that simple functions are dense in $L^\infty(\mu)$ and use this fact.*

Problem 5. Let Γ be a set containing at least two points.

(1) Describe $\operatorname{ext} B_{\ell^1(\Gamma,\mathbb{R})}$.
(2) Describe $\operatorname{ext} B_{\ell^1(\Gamma,\mathbb{C})}$.
(3) Is $\overline{\operatorname{co}\operatorname{ext} B_{\ell^1(\Gamma,\mathbb{F})}}^{\|\cdot\|} = B_{\ell^1(\Gamma,\mathbb{F})}$?

Problem 6. Let K be a compact Hausdorff space.

(1) Describe $\operatorname{ext} B_{\mathcal{C}(K,\mathbb{R})^*}$.
(2) Describe $\overline{\operatorname{co}\operatorname{ext} B_{\mathcal{C}(K,\mathbb{R})^*}}^{\|\cdot\|}$.
(3) Show that $\overline{\operatorname{co}\operatorname{ext} B_{\mathcal{C}(K,\mathbb{R})^*}}^{w^*} = B_{\mathcal{C}(K,\mathbb{R})^*}$ without using Krein-Milman theorem.
(4) Solve problems (1)–(3) for $\mathcal{C}(K, \mathbb{C})$.

`Hint:` *Use the Riesz representation theorem to represent $\mathcal{C}(K, \mathbb{F})^*$ as a space of measures. (1) In the real case show that extreme points are just $\pm\delta_x$, $x \in K$. In the complex case show that extreme points are multiples of Dirac measures by a complex unit. (2) Show that the set consist exactly of measures supported by a countable set. (3) Use (1) an the bipolar theorem.*

Problem 7. Let X be a Banach space

(1) Show that $\overline{\operatorname{co}\operatorname{ext} B_{X^*}}^{w^*} = B_{X^*}$.
(2) Suppose that X is reflexive. Show that $\overline{\operatorname{co}\operatorname{ext} B_{X^*}}^{\|\cdot\|} = B_{X^*}$.
(3) Show by examples that for a nonreflexive space one can have either $\overline{\operatorname{co}\operatorname{ext} B_{X^*}}^{\|\cdot\|} = B_{X^*}$ or $\overline{\operatorname{co}\operatorname{ext} B_{X^*}}^{\|\cdot\|} \subsetneqq B_{X^*}$ and both possibilities can take place.

`Hint:` *(1) Combine Krein-Milman and Banach-Alaoglu theorems. (3) Use some of the preceding problems.*

Problem 8. Let K be a compact Hausdorff space.

(1) Describe $\operatorname{ext} B_{\mathcal{C}(K,\mathbb{F})}$.
(2) Deduce that in case K is connected and contains at least two points the space $\mathcal{C}(K,\mathbb{R})$ is not isometric to a dual Banach space.

Problem 9. Show that $\operatorname{ext} B_{L^1([0,1])} = \emptyset$.

Problem 10. Show that $\operatorname{ext} B_{c_0} = \emptyset$.

Problem 11. Let K be a compact convex subset of a HLCS. The point $x \in K$ is called an **exposed point** of K if there is a continuous affine function $f : K \to \mathbb{R}$ such that $f(y) < f(x)$ for each $y \in K \setminus \{x\}$.

(1) Show that any exposed point is also an extreme point.
(2) Show that an extreme point need not be an exposed point.

`Hint:` *(2) Consider the set* $K = \operatorname{co}(B((0,0),1) \cup B((1,0),1))$ *in* $\mathbb{R}^2$.

Problem 12. Let K be a compact convex subset of a HLCS. A subset $F \subset K$ is called an **exposed face** of K if there is a continuous affine function $f : K \to \mathbb{R}$ such that $F = \{x \in K; f(x) = \max f(K)\}$.

(1) Show that any exposed face of K is also a closed face of K.
(2) Show that a closed face of K need not be an exposed face.
(3) Let F_1 be an exposed face of K and let F_2 be an exposed face of K. Is F_2 necessarily an exposed face of K?

`Hint:` *(2) Use Problem 11. (3) Consider the example from Problem 11.*

Problem 13. Let K be a compact convex subset of a HLCS and $\mu = \sum_{j=1}^{n} t_j \delta_{x_j}$ a finitely supported probability measure on K (i.e., $x_1, \dots, x_n \in K$, $t_1, \dots, t_n \in [0,1]$, $t_1 + \dots + t_n = 1$). Find the barycenter of μ.

Problem 14. Let $K = [0,1]$ and let λ be the Lebesgue measure on $[0,1]$. Find the barycenter of λ.

Problem 15. Let $K \subset \mathbb{R}^2$ be a nondegenerate triangle with vertices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. Let

$$\mu = \frac{\lambda^2|_K}{\lambda^2(K)},$$

where λ^2 is the two-dimensional Lebesgue measure. Show that the barycenter of μ coincides with the geometric barycenter of the triangle K (i.e., with $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$).

`Hint:` *Since the Lebesgue measure is invariant with respect to translation and rotation, suppose without loss of generality that* $\boldsymbol{a} = (0,0)$, $\boldsymbol{b} = (b,0)$ *and* $\boldsymbol{c} = (c_1, c_2)$. *Then use the definitions and Fubini theorem.*

Problem 16. Let μ be a Borel probability measure on $[0,1]$. Find a formula for its barycenter.

Problem 17. Let $K \subset \mathbb{R}^n$ be a compact convex set and let μ be a Borel probability measure on K. Find a formula for its barycenter.

`Hint:` *Apply the definition to coordinate projections.*

Problem 18. Let

$$K = \{(x,y,z) \in \mathbb{R}^3; x^2 + y^2 \leq 1 \ \& \ z \in [-1,1]\}.$$

Show that K is a compact convex set and describe $\operatorname{ext} K$.

Problem 19. Let $A = \{(x,y,0) \in \mathbb{R}^3; x^2+y^2 \leq 1\}$ and $K = \operatorname{co}(A \cup \{(1,0,1),(1,0,-1)\})$. Show that K is a compact convex set, describe $\operatorname{ext} A$ and show that $\operatorname{ext} A$ is not closed.

Problem 20. Let $\varphi : [0,1] \to (0,+\infty)$ be a continuous concave function. Let

$$K = \{(x,y,z) \in \mathbb{R}^3; z \in [0,1] \ \& \ x^2 + \frac{y^2}{\varphi(z)} \leq 1\}.$$

(1) Show that K is a compact convex set.
(2) Describe $\operatorname{ext} K$.
(3) Assume that φ is not affine on $[0,1]$. Show that $\operatorname{ext} K$ is not closed in K.

`Hint:` *(1) Show that the function $(x,y,z) \mapsto x^2 + \frac{y^2}{\varphi(z)}$ is convex. (2) First show that all the extreme points are on the boundary of K and that the boundary of K is the union of closed discs $D_0 = \{(x,y,0); x^2+\frac{y^2}{\varphi(0)} \leq 1\}$, $D_1 = \{(x,y,1); x^2+\frac{y^2}{\varphi(1)} \leq 1\}$ and the set $B = \{(x,y,z) \in \mathbb{R}^3; z \in [0,1] \ \& \ x^2 + \frac{y^2}{\varphi(z)} = 1\}$. Further show that a boundary point of K is not an extreme point of K if and only if it is the center of a nondegenerate segment contained in the boundary of K and that any such segment is contained either in D_0 or in D_1 or in B. Deduce that the extreme points contained in D_0 or D_1 are exactly the points of boundary circles of these discs. Finally suppose that (x_1,y_1,z_1) and (x_2,y_2,z_2) are two different points in B such that the segment connecting them is contained in B. Show that $x_1 = y_1$ and $y_1 = y_2$, so we can suppose we have points (x,y,z_1) and (x,y,z_2). In case $y \neq 0$ show that φ is affine on the interval $[z_1,z_2]$. Summarize these results to get a description of $\operatorname{ext} K$. (3) If φ is not affine, then there is $z_0 \in (0,1)$ such that the point $(z_0,\varphi(z_0))$ is not the center of any nondegenerate segment on the graph of φ. Show that $(1,0,z_0) \in \overline{\operatorname{ext} K} \setminus \operatorname{ext} K$.*

Problem 21. Let $\psi : [0,2\pi] \to [0,\infty)$ be a bounded upper semicontinuous function such that $\psi(0) = \psi(2\pi)$. (Recall that ψ is **upper semicontinuous** if $\{t; \psi(t) < c\}$ is open for eah $c \in \mathbb{R}$.) Set

$$A = \{(\cos t, \sin t, z); t \in [0,2\pi] \ \& \ |z| \leq \psi(t)\}$$

and $K = \operatorname{co} A$.

(1) Show that K is a compact convex set.
(2) Describe $\operatorname{ext} K$.
(3) Suppose that $\psi(t) = R(\frac{t}{2\pi})$, where R is the Riemann function, i.e.

$$R(t) = \begin{cases} \frac{1}{q} & \text{if } t = \frac{p}{q} \text{ where } p \in \mathbb{Z},\ q \in \mathbb{N} \text{ and } p,q \text{ are mutually prime,} \\ 0 & \text{if } t \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that $\operatorname{ext} K$ is a G_δ set which is not F_σ.

`Hint:` *(1) Show that A is compact by showing it is closed and bounded and use the fact that in a finite-dimensional space the convex hull of a compact set is again compact. (2) Using Proposition IX.6 show that* $\operatorname{ext} K$ *consists exactly of those elements of A which are not the center of a nondegenerate segment contained in A. (3) Let $B = \{(\cos t, \sin t, 0); t \in [0, 2\pi]\}$. Show that B is a closed subset of K such that both sets $B \cap \operatorname{ext} K$ and $B \setminus \operatorname{ext} K$ are dense in B. Since* $\operatorname{ext} K$ *is G_δ by Proposition IX.9(a), use Baire category theorem to show that* $\operatorname{ext} K$ *is not F_σ.*

Problem 22. Let K be a compact Hausdorff space and let $P(K)$ denote the set of all the Radon probability measures on K equipped with the weak* topology inherited from $\mathcal{C}(K)^*$. Denote by $\delta : K \to P(K)$ the mapping assigning to each $x \in K$ the Dirac measure δ_x supported at x. For any $f \in \mathcal{C}(K)$ denote by $\tilde{f}$ the function on $P(K)$ defined by

$$\tilde{f}(\mu) = \int f \, \mathrm{d}\mu, \quad \mu \in P(K).$$

(1) Show that, given $f \in \mathcal{C}(K)$, the function $\tilde{f}$ is a continuous affine function on $P(K)$.
(2) Let $\mu \in P(K)$. Denote by $\delta(\mu)$ the image of μ by the mapping δ. Show that $\delta(\mu)$ is a Radon probability measure on $P(K)$ and its barycenter is μ.
(3) Show that the mapping $f \mapsto \tilde{f}$ is a linear isometry of $\mathcal{C}(K)$ onto the space of all affine continuous functions on $P(K)$ equipped with the sup-norm.

Problem 23. Let K be a compact convex subset of a HLCS and let $f : K \to \mathbb{R}$ be a continuous affine function. Show that f attains its maximum on K at some extreme point of K.

`Hint:` *The set $\{x \in K; f(x) = \max f(K)\}$ is a closed face of K.*