

Lemma 1.25.

E ... abstract spectral measure
in a Hilbert space H

\mathcal{A} ... domain σ -algebra of E

For $x, y \in H$... $E_{x,y}(A) = \langle E(A)x, y \rangle$, $A \in \mathcal{A}$

(a) $x \mapsto E_{x,y}$ is linear
(s) $y \mapsto E_{x,y}$ is conjugate linear

} clear, using properties of the inner product

(c) $E_{x,y} = \overline{E_{y,x}}$

$A \in \mathcal{A} \Rightarrow \langle E_{x,y}(A) = \langle E(A)x, y \rangle = \langle x, E(A)y \rangle = \overline{\langle E(A)y, x \rangle} = \overline{E_{y,x}(A)}$

$E(A)$ is a OS projection, hence $E(A)^* = E(A)$

(d) $E_{x,x} \geq 0$ $\Gamma E_{x,x}(A) = \langle E(A)x, x \rangle = \|E(A)x\|^2 \geq 0$
by properties of OS projections

(e) $E_{x,y} = \frac{1}{4} (E_{x+y, x+y} - E_{x-y, x-y} + i E_{x+iy, x+iy} - i E_{x-iy, x-iy})$

Γ polarization identity - direct computation using (a), (s)

(f) $\forall x, y \in H \forall A \in \mathcal{A} : |E_{x,y}(A)| \leq \sqrt{E_{x,x}(A) E_{y,y}(A)} \leq \frac{1}{2} (E_{x,x}(A) + E_{y,y}(A))$

$\Gamma B(x,y) := E_{x,y}(A)$ by (a, s, c, d) B is

a "semi-inner product", i.e. positive semi-definite hermitian sesquilinear form. Such forms satisfy Cauchy-Schwarz inequality (with the same proof as for an inner product), i.e.

$|B(x,y)| \leq \sqrt{B(x,x) B(y,y)}$, which is the first inequality

The second one is just the inequality between the geometric and arithmetic means

Remark to (f): It follows that $|E_{x,y}| \leq \frac{1}{2}(E_{x,x} + E_{y,y})$

$$|E_{x,y}|(A) = \sup \left\{ \sum_1^n |E_{x,y}(B_j)| : B_j \text{ disjoint}, B_j \in \mathcal{A}, \bigcup_{j=1}^n B_j = A \right\}$$

$$\leq \sup \left\{ \sum_1^n \frac{1}{2}(E_{x,x}(B_j) + E_{y,y}(B_j)) : \dots \right\} = \frac{1}{2}(E_{x,x}(A) + E_{y,y}(A))$$

(g) $E_{x+y,x+y} \leq 2(E_{x,x} + E_{y,y})$

$$\Gamma E_{x+y,x+y}(A) \stackrel{(a),(b)}{=} E_{x,x}(A) + E_{x,y}(A) + E_{y,x}(A) + E_{y,y}(A)$$

$$\stackrel{(c)}{=} E_{x,x}(A) + 2 \operatorname{Re} E_{x,y}(A) + E_{y,y}(A) \stackrel{(f)}{\leq} 2(E_{x,x}(A) + E_{y,y}(A))$$

(h) $\|E_{x,y}\| \leq \|x\| \cdot \|y\|$

Recall $\|E_{x,y}\| = |E_{x,y}|(\mathcal{E}) = \sup \left\{ \sum_1^n |E_{x,y}(A_j)| : A_j \in \mathcal{A} \text{ disjoint} \right\}$

So, fix $A_1, \dots, A_n \in \mathcal{A}$ disjoint and compute:

$$\sum_1^n |E_{x,y}(A_j)| = \sum_1^n |\langle E(A_j)x, y \rangle| = \sum_1^n d_j \cdot \langle E(A_j)x, y \rangle =$$

d_1, \dots, d_n suitable complex units

$$\stackrel{(a)}{=} \left\langle \sum_1^n d_j E(A_j)x, y \right\rangle \leq \left\| \sum_1^n d_j E(A_j)x \right\| \cdot \|y\| =$$

A_1, \dots, A_n disjoint $\Rightarrow E(A_j)x, j=1, \dots, n$ mutually orthogonal by (vi) in the definition of abstract sp. meas.

$$\leq \left(\sum_{j=1}^n \|d_j E(A_j)x\|^2 \right)^{1/2} \cdot \|y\| =$$

$$\stackrel{\uparrow |d_j|=1}{=} \left(\sum_{j=1}^n \|E(A_j)x\|^2 \right)^{1/2} \|y\| = \left\| \sum_{j=1}^n E(A_j)x \right\| \|y\| =$$

$$= \left\| E\left(\bigcup_{j=1}^n A_j\right)x \right\| \|y\| \leq \|x\| \cdot \|y\|$$

\uparrow
(vi)