Proof of Theorem X.29

(a) $\Phi(f) + \Phi(g) \subset \Phi(f+g)$

$\ulcorner x \in D(\Phi(f) + \Phi(g)) = D(\Phi(f)) \cap D(\Phi(g))$

$\Rightarrow f, g \in L^2(\tilde{E}_{x,x}) \Rightarrow f+g \in L^2(\tilde{E}_{x,x}) \Rightarrow x \in D(\Phi(f+g))$

Moreover, for $x, y \in D(\Phi(f) + \Phi(g))$ we have

$\langle (\Phi(f) + \Phi(g))x, y \rangle = \langle \Phi(f)x, y \rangle + \langle \Phi(g)x, y \rangle =$

$= \int f \, d\tilde{E}_{x,y} + \int g \, d\tilde{E}_{x,y} = \int (f+g) \, d\tilde{E}_{x,y} = \langle \Phi(f+g)x, y \rangle$

Since $D(\Phi(f)) \cap D(\Phi(g)) = D(\Phi(|f|+|g|))$ is dense,

we deduce $\cancel{D \subset} \Phi(f)x + \Phi(g)x = \Phi(f+g)x$ $\lrcorner$

(b) $\Phi(f)\,\Phi(g) \subset \Phi(f \cdot g)$ & $D(\Phi(f)\,\Phi(g)) = D(\Phi(g)) \cap D(\Phi(f \cdot g))$

$\ulcorner x \in D(\Phi(f)\,\Phi(g)) \Leftrightarrow x \in D(\Phi(g))$ & $\Phi(g)x \in D(\Phi(f))$

$\Leftrightarrow g \in L^2(\tilde{E}_{x,x})$ & $f \in L^2(\tilde{E}_{\Phi(g)x, \Phi(g)x}) \Leftrightarrow (*)$

Compute $E_{\Phi(g)x, \Phi(g)x}(A) = \langle E(A)\Phi(g)x, \Phi(g)x \rangle =$

$\underset{\underset{E(A) \text{ is an OS projection}}{\uparrow}}{=} \|E(A)\Phi(g)x\|^2 = \underset{\underset{g_n \text{ as in Proof of Thm 28 } ⑤}{\uparrow}}{\lim_{n \to \infty}} \|E(A)\Phi_0(g_n)x\|^2 =$

$= \underset{\underset{\text{Thm 27(a)}}{\uparrow}}{\lim_{n \to \infty}} \|\Phi_0(\chi_A g_n)\|^2 = \underset{\underset{\text{Thm 27(d)}}{\uparrow}}{\lim_{n \to \infty}} \int_A |g_n|^2 \, d\tilde{E}_{x,x} = \int_A |g|^2 \, d\tilde{E}_{x,x}$

So, $E_{\underline{\Phi}(g)_+, \underline{\Phi}(g)_+}$ is absolutely continuous with respect to $E_{+,+}$, with density $|g|^2$

So, $\int |f|^2 \, dE_{\underline{\Phi}(g)_+, \underline{\Phi}(g)_+} = \int |fg|^2 \, dE_{+,+}$

Hence, continue:

$(*) \Leftrightarrow g \in L^2(\hat{E}_{+,+}) \ \& \ fg \in L^2(\hat{E}_{+,+}) \Leftrightarrow x \in D(\underline{\Phi}(g)) \cap D(\underline{\Phi}(fg))$

The equality $\underline{\Phi}(fg)_+ = \underline{\Phi}(f)\underline{\Phi}(g)_+$:

- f, g bdd $\cdots$ by Thm 27 (a)

- f bdd, g not necessarily $\cdots$ take g_n as in proof of Thm 28 ⑤

  The $\underline{\Phi}(g)x = \lim\limits_{n \to \infty} \underline{\Phi}_0(g_n)x$, $+\in D(\underline{\Phi}(g))$)

  f bdd $\Rightarrow \underline{\Phi}(f) \in L(H)$, so

  $\Rightarrow \underline{\Phi}(f)\underline{\Phi}(g)_+ = \lim\limits_{n \to \infty} \underline{\Phi}(f)\underline{\Phi}_0(g_n)_+ =$

  $= \lim\limits_{n \to \infty} \underline{\Phi}_0(fg_n)_+ = \underline{\Phi}(fg)_+$

  $\quad\quad\quad\quad \uparrow (*)$
  $\left(fg_n \to fg \text{ in } L^2(\hat{E}_{+,+}) \right)$

- f, g general $\cdots$ take f_n as in proof of Thm 28 ⑤

  $\Rightarrow \underline{\Phi}(f)\underline{\Phi}(g)x = \lim\limits_{n \to \infty} \underline{\Phi}_0(f_n)\underline{\Phi}(g)_+ = \lim\limits_{n \to \infty} \underline{\Phi}_0(f_n g)_+ \xrightarrow{(*)}$

  $= \underline{\Phi}(fg)x$

In $(\#)$ we use Lebesgue dominated convergence theorem

$$\langle \overline{\Phi}_0 (f g_n) x, y \rangle = \int f g_n \, dE_{x,y} \longrightarrow \int f g \, dE_{x,y}$$

$|fg|$ is an integrable majorant.

$$\int f g \, dE_{x,y} = \langle \overline{\Phi}(fg) x, y \rangle$$

(C) $\overline{\Phi}(f)^* = \overline{\Phi}(\overline{f})$, $\overline{\Phi}(f) \overline{\Phi}(f)^* = \overline{\Phi}(f)^* \overline{\Phi}(f) = \overline{\Phi}(|f|^2)$

⌐ • $D(\overline{\Phi}(\overline{f})) = D(\overline{\Phi}(f))$ .... clear , as $|\overline{f}|^2 = |f|^2$

• $x, y \in D(\overline{\Phi}(f)) \implies$

$$\langle \overline{\Phi}(f) x, y \rangle = \lim_n \langle \overline{\Phi}_0 (f_n) x, y \rangle = \lim_n \langle x, \overline{\Phi}_0 (f_n)^* y \rangle$$

$$= \lim_n \langle x, \overline{\Phi}_0 (\overline{f_n}) y \rangle = \langle x, \overline{\Phi}(\overline{f}) y \rangle$$

So, $\overline{\Phi}(\overline{f}) \subset \overline{\Phi}(f)^*$

• Conversely. Let $A_n := \{ \lambda ; |f(\lambda)| \le n \}$. Then

$$E(A_n) \overline{\Phi}(f)^* \subset (\overline{\Phi}(f) E(A_n))^* = (\overline{\Phi}(f) \overline{\Phi}_0(\chi_{A_n}))^*$$

$$\overset{(5)}{=} (\overline{\Phi}(f \chi_{A_n}))^* = \overline{\Phi}(\overline{f} \chi_{A_n})$$

$x \in D(\overline{\Phi}(f)^*) \implies E(A_n) \overline{\Phi}(f)^* x \longrightarrow \overline{\Phi}(f)^* x$

$\implies \overline{\Phi}(f)^* x = \lim_{n \to \infty} \overline{\Phi}(\overline{f} \chi_{A_n}) x$

$$\Rightarrow \overset{\infty}{\underset{}{}} \langle \Phi(f)^*_{,+}\rangle = \lim_{u\to\infty} \langle \hat{\Phi}(\bar{f}\,\chi_{A_n})_{,+}\rangle = \lim_{n\to\infty} \int |\bar{f}\,\chi_{A_n}|^2 \, dE_{t_{,+}} =$$

$$= \int |f|^2 \, dE_{t_{,+}} \quad , \quad so \quad +\in D(\hat{\Phi}(f)).$$

Hence $D(\Phi(f)^*) \subset D(\Phi(f))$. IT follows that $\bar{\Phi}(f)^* = \Phi(\bar{f})$

- $\bar{\Phi}(f)^* \bar{\Phi}(f) = \bar{\Phi}(f)\bar{\Phi}(f)^* \subset \bar{\Phi}(|f|^2)$ by (3)

$$\ulcorner D(\Phi(f)^* \Phi(f)) = D(\Phi(f)) \cap D(\Phi(|f|^2))$$
$$D(\hat{\Phi}(f) \Phi(f)^*) = D(\Phi(\bar{f})) \cap D(\Phi(|f|^2)) \quad \lrcorner$$

- $D(\Phi(|f|^2)) \subset D(\Phi(f))$

$$\ulcorner +\in D(\Phi(|f|^2)) \Rightarrow |f|^2 \in L^2(E_{+,+}) \Rightarrow \int |f|^2 \in L^1(E_{t_{,}+})$$
$$\Rightarrow f \in L^2(E_{x_{,}+}) , \quad so \quad +\in D(\Phi(f)) \rangle \quad \lrcorner$$

(d) $\Phi(f)$ is a closed operator

$$\ulcorner \Phi(f) = \Phi(\bar{f})^* \quad \lrcorner$$

(e) $\Phi(f)$ is cts $\iff$ f is essentially bdd

$$\ulcorner \Leftarrow : \text{by Th 27}$$

$$\Rightarrow : \Phi(f) \text{ bdd} \Rightarrow D(\Phi(f)) = H \quad , \quad f_n \text{ as above}$$
$$(i.e. \quad f_n = f \cdot \chi_{A_n}, \quad A_n = \{\lambda, |f(\lambda)| \le n\}$$
$$\Rightarrow \bar{\Phi}_0(f_n)+ \longrightarrow \Phi(f)+ \quad , +\in H|$$

uniform bddness principle $\Rightarrow (\Phi_0(f_n))$ is uniformly bdd

but $\|\Phi_0(f_n)\| = \|f_n^*\|_\infty \Rightarrow f \text{ ess. bdd} \quad \lrcorner$