

Proposition X.32

$$T = \int g dE \quad (g \text{ s.d.d.})$$

$\Rightarrow$  the spectral measure of  $T$  is

$$E_T(A) = E(g^{-1}(A)), \quad A \in \mathcal{A}_T = \{A \subset \mathbb{C} : g^{-1}(A) \in \mathcal{A}\}$$

Proof: Thm 26(e)  $\Rightarrow \forall f \in \mathcal{C}(\sigma(T)) :$

$$\widehat{f}(T) = \int f \circ g dE$$

So, for  $x, y \in H$  and  $f \in \mathcal{C}(\sigma(T)) :$

$$\begin{aligned} \langle \widehat{f}(T)x, y \rangle &= \langle \left( \int f \circ g dE \right) x, y \rangle = \int (f \circ g) dE_{x,y} = \\ &= \int f d g(E_{x,y}) \end{aligned}$$

So,  $(E_T)_{x,y} = g(E_{x,y})$ , i.e.

$$(E_T)_{x,y}(A) = E_{x,y}(g^{-1}(A)).$$

This shows that  $\mathcal{A}_T$  is as above. Moreover,

$$\begin{aligned} \langle E_T(A)x, y \rangle &= \langle \widehat{\chi_A}(T)x, y \rangle = \int \chi_A d g(E_{x,y}) = \\ &= \int \chi_A \circ g dE_{x,y} = \int \chi_{g^{-1}(A)} dE_{x,y} = E_{x,y}(g^{-1}(A)) = \\ &= \langle E(g^{-1}(A))x, y \rangle \end{aligned}$$

Theorem 1.33  $T \in L(H)$  s.d.d

(a)  $T = \int \text{cd} dE_T$  . Moreover  $T = \int \text{cd} dE \Rightarrow E = E_T$

$$\begin{aligned} \Gamma \langle (\int \text{cd} dE_T) x, y \rangle &= \int \text{cd} d(E_T)_{x,y} = \\ &= \langle \tilde{\text{cd}}(T)_{+10} \rangle = \langle T_{+10} \rangle \end{aligned}$$

The uniqueness follows from Prop. 32

(b)  $g$  s.d.d  $\mathcal{A}$ -measurable  $\Rightarrow \tilde{g}(T) = \int g dE_T$

$$\Gamma \langle (\int g dE_T)_{+10} \rangle = \int g d(E_T)_{+10} = \langle \tilde{g}(T)_{+10} \rangle$$