Proof of Lemma X.38

Let T be a closely defined closed operator

(1) $I + T^*T$ is one-to-one

$\ulcorner$ Let $x \in D(I+T^*T) = D(T^*T)$ be such that $(I+T^*T)x = 0$

Then $0 = \langle (I+T^*T)x, x\rangle = \langle x,x\rangle + \langle T^*Tx, x\rangle \overset{x\in D(T),\, Tx\in D(T^*)}{=}$

$= \langle x,x\rangle + \langle Tx,Tx\rangle \geq \langle x,x\rangle \Rightarrow x = 0$ $\lrcorner$

(2) Recall that $G(T^*) = V(G(T)^\perp)$, where $V(x,y) = (-y,x)$ (by Lemma 13)

So, $G(T)^\perp = V(G(T^*))$

Let P be the OG projection of $H \oplus H$ onto $G(T)$
Define $B, C \in \mathcal{L}(H)$ by

$$Bu = \pi_1 P(u,0) \qquad u \in H$$
$$Cu = -\pi_2 (I-P)(u,0)$$

(where $\pi_1(x,y) = x$, $\pi_2(x,y) = y$)

Clearly $B, C \in \mathcal{L}(H)$, $\|B\| \leq 1$, $\|C\| \leq 1$

Further, $P(u,0) = (Bu, TBu)$ $\quad (R(P) = G(T))$

$(I-P)(u,0) = (T^*Cu, -Cu)$

(as $R(I-P) = \operatorname{Ker} P = G(T)^\perp =$
$= V(G(T^*))$)

So, $(u,0) = P(u,0) + (I-P)(u,0) = (Bu + T^*Cu, TBu - Cu)$
for $u \in H$

It follows $TBu - Cu = 0$, so $-C = TB$

$Bu + T^*Cu = u$, so $I = B + T^*C = B + T^*TB$
$= (I + T^*T)B$

So, $I = (I+T^*T)B$. Hence, $R(I+T^*T) = H$.

As $I+T^*T$ is one-to-one (by ①),

$R(B) = D(I+T^*T) = D(T^*T)$

③ Conclusion from ① and ②: $I+T^*T$ is one-to-one and onto,

$B = (I+T^*T)^{-1}$, $C = TB$, $\|B\| \le 1$, $\|C\| \le 1$

④ $B \ge 0$ [Hence, (a) and (b) hold]

[The computation in ① yields $\langle (I+T^*T)x, x\rangle \ge 0$, $x \in D(T^*T)$

So, for $u \in H$: $\langle Bu, u\rangle = \langle Bu, (I+T^*T)Bu\rangle \ge 0$]

⑤ $D(T^*T)$ is dense in H

[$D(T^*T) = R(B)$, $B \ge 0$, so B is selfadjoint.

B is one-to-one (being an inverse), so $R(B)$ is dense]

(by Prop. 12)

⑥ T^*T is self-adjoint

[$(I+T^*T) = B^{-1} \Rightarrow I+T^*T$ is self-adjoint by Prop. 17(e)

so T^*T is self-adjoint as well (Prop. 11(b)]

⑦ $T = \overline{T\restriction_{D(T^*T)}}$

[i.e., $G(T\restriction D(T^*T))$ is dense in $G(T)$. If not, then $\exists (x, Tx) \in G(T)$

$(x, Tx) \perp G(T\restriction D(T^*T)) = G(T\restriction R(B)) \Rightarrow \forall u \in H: (x, Tx) \perp (Bu, TBu)$

So, $0 = \langle x, Bu\rangle + \langle Tx, TBu\rangle = \langle x, Bu\rangle + \langle x, T^*TBu\rangle = \langle x, Bu + T^*TBu\rangle$

$Bu \in D(T^*T) \Rightarrow TBu \in D(T^*)$ $\qquad = \langle x, u\rangle$

So, $x = 0$]