

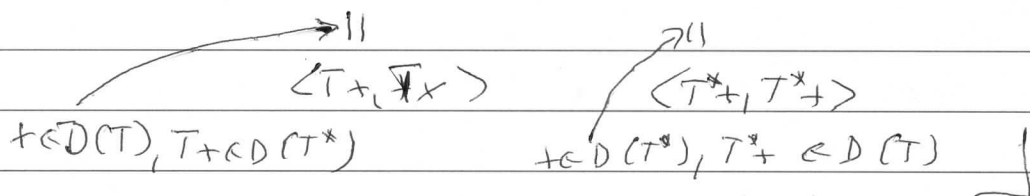
Proof of Lemma X.39

Let T be a normal operator on H , i.e. $T^*T = TT^*$

(1) $\|T_+ \| = \|T^*_+ \|$, $x \in D(T^*T) = D(TT^*)$

$T^*T = TT^* \Rightarrow D(T^*T) = D(TT^*)$

$x \in D(T^*T) \Rightarrow \langle T^*T_+ x, x \rangle = \langle TT^*_+ x, x \rangle$



(2) $D(T) \subset D(T^*)$, $\|T^*_+ \| \leq \|T_+ \|$, $x \in D(T)$

$x \in D(T) \xrightarrow{\text{L.S.C.}} \exists (x_n) \subset D(T^*T) : x_n \rightarrow x \ \& \ T x_n \rightarrow T x$

$y \in D(T) \Rightarrow \langle T y, x \rangle = \lim_n \langle T y, x_n \rangle = \lim_n \langle y, T^*_+ x_n \rangle$
 \uparrow
 $x_n \in D(T^*T) = D(TT^*) \subset D(T^*)$

$\Rightarrow |\langle T y, x \rangle| \leq \limsup | \langle y, T^*_+ x_n \rangle | \leq \|y\| \cdot \limsup \|T^*_+ x_n\| \stackrel{(1)}{=} \|y\| \limsup \|T x_n\| = \|y\| \cdot \|T_+ \|$

$\Rightarrow y \mapsto \langle T y, x \rangle$ is cts on $D(T)$, i.e. $x \in D(T^*)$

Moreover, the computation shows that $\|T^*_+ \| \leq \|T_+ \|$

(3) As T^* is also normal, by (2) it follows $D(T^*) = D(T)$, $\|T_+ \| = \|T^*_+ \|$, $x \in D(T)$, so (a), (b) is proved

(4) $S \supset T$, S normal $\Rightarrow S^* \subset T^*$

So $D(T) \subset D(S) = D(S^*) \subset D(T^*) = D(T)$
 $\Rightarrow D(T) = D(S)$