Proof of Theorem X.40 Let T be a normal operator on H

(1) Let $B, C \in \mathcal{L}(H)$ be as in Lemma 38
Then $BT \subset TB$ & $BC = CB$

$\Big[BT = BT(I+T^*T)B = B(T+TT^*T)B =$

$(I+T^*T)B = I$ $\qquad$ $\supset$ by Prop. 3 (iii)

Moreover, $D(T(I+T^*T)) =$
$= \{x \in D(T^*T),\ x+T^*Tx \in D(T)\} =$
$= \{x \in D(T^*T),\ T^*Tx \in D(T)\} =$
$= D(TT^*T) = D(TT^*T) \cap D(T)$

$T^*T = TT^*$

$= B(T+T^*TT)B = \underbrace{B(I+T^*T)}_{\subset I} TB \subset TB$

we already know $BT \subset TB$

$BC = BTB \subset TBB = CB$

But $BC, CB \in \mathcal{L}(H)$, so $BC = CB \Big]$

(2) Recall: $B \geq 0$, $\|B\| \leq 1 \Rightarrow \sigma(B) \subset [0,1]$,
0 is not an eigenvalue (B is one-to-one), so $E_B(\{0\}) = 0$

Set $P_j := \widehat{\chi_{(\frac{1}{j+1}, \frac{1}{j}]}}(B)$ (measurable calculus)

$S_j := \widehat{\left(\frac{1}{\varphi} \chi_{(\frac{1}{j+1}, \frac{1}{j}]}\right)}(B)$, where $\varphi(t) = t$

Then $P_j, S_j \in \mathcal{L}(H)$, commute with each other and with B, P_j are OG projections
P_j mutually orthogonal
all these operators commute with C (as $CB = BC$)
$P_j = S_j B = B S_j$ $\quad$ ($B = \widehat{id}(B) = \widehat{\varphi}(B)$)

(3) $\sum_{j=1}^{\infty} P_j = I$

$\ulcorner$ $\langle (\sum_{j=1}^{\infty} P_j) x, y \rangle = \lim_{n\to\infty} \langle (\sum_{j=1}^{n} P_j) x, y \rangle =$

$= \lim_{n\to\infty} \langle \widetilde{\chi_{(\frac{1}{n+1}, 1]}}(B) x, y \rangle = \langle \widetilde{\chi_{(0,1]}} x, y \rangle =$

$= \langle \widetilde{1} x, y \rangle = \langle I x, y \rangle = \langle x, y \rangle$

$\chi_{(0,1]} = 1$ E_B - a.e., as $\sigma(B) \subset [0,1]$, $E_B(\{0\}) = 0$ $\lrcorner$

(4) $TP_j \in \mathcal{L}(H)$, $P_j T \subset T P_j$

$\ulcorner$ $TP_j = TBS_j = CS_j \in \mathcal{L}(H)$

$P_j T = S_j BT \underset{(1)}{\subset} S_j TB = S_j C = C S_j = TP_j$ $\lrcorner$

(5) TP_j is a normal operator

$\ulcorner$ $(TP_j)^* \overset{(4)}{\subset} (P_j T)^* \underset{\text{Prop. 11(c)}}{=} T^* P_j \Rightarrow (TP_j)^* = T^* P_j$

as $TP_j \in \mathcal{L}(H)$

So, $\forall x \in H$: $\|(TP_j)^* x\| = \|T^* P_j x\| \underset{L\,39(b),\ P_j x \in D(T)}{=} \|TP_j x\|$

as $TP_j \in \mathcal{L}(H)$

So, TP_j is normal, by a proposition from IX.1 $\lrcorner$

(6) Let E_j be the spectral measure of TP_j ($\mathcal{A}_j$ the σ-algebra)

Then $E_j(A)$ commutes with P_j, $A \in \mathcal{A}_j$

$\ulcorner$ enough to observe: P_j commutes with TP_j

$P_j TP_j \underset{(4)}{\subset} TP_j P_j$ Since P_j and $TP_j \in \mathcal{L}(H)$,

necessarily $P_j TP_j = TP_j P_j$ $\lrcorner$

⑦ Let $E := \sum_{j=1}^{\infty} E_j \cdot P_j$, i.e.

$$E(A) = \sum_{j=1}^{\infty} E_j(A) P_j, \quad A \in \mathcal{A} = \bigcap_{j=1}^{\infty} \mathcal{A}_j$$

Then E is a well-defined spectral measure

- $E_j(A) P_j = P_j E_j(A) \Rightarrow E_j(A) P_j$ is an OS projection

~~so~~
Moreover, as P_j are mutually OS, also these are mutually OS, hence their sum is an OS projection.

So (i) and (ii) hold

(iii): $E(\emptyset) = 0$ – clear

$E(\mathbb{C}) = I$: $E(\mathbb{C}) = \sum_j E_j(\mathbb{C}) P_j = \sum_j P_j = I$ by ③

(iv) .. clear

(v) $E(A \cap B) = \sum_j E_j(A \cap B) P_j = \sum_j E_j(A) E_j(B) P_j$

$E(A)E(B) = E(A) \sum_j E_j(B) P_j = \sum_j E(A) P_j E_j(B) =$

$= \sum_j E_j(A) P_j E_j(B) = \sum_j E_j(A) E_j(B) P_j$

$E(A)P_j = \left(\sum_k E_k(A) P_k\right) P_j = E_j(A) P_j$

(vi) clear

(vii) $E_{x,x}(A) = \langle E(A)x, x\rangle = \sum_j \langle E_j(A)P_j x, x\rangle =$

$= \sum_j \langle E_j(A) P_j x, P_j x\rangle = \sum_j E_{P_j x, P_j x}(A)$

$\Rightarrow E_{x,x} = \sum_j E_{P_j x, P_j x}$, so it will work.

(8) $T = \int id\, dE$. By L 39 (c) it is enough to show "$\subset$"

$x \in D(T) \Rightarrow \int |z|^2 dE_{x,x} = \sum_j \int |z|^2 d(E_j)_{P_j x, P_j x} =$

$= \sum_j \|TP_j x\|^2 = \sum_j \|P_j T x\|^2 \underset{(3)}{=} \|Tx\|^2 \Rightarrow x \in D\left(\int id\, dE\right)$

Thm 27 (d) $\qquad x \in D(T),\ P_j T \subset TP_j$

$x, y \in D(T)$:

$\langle (\int id\, dE) x, y\rangle = \int id\, dE_{x,y} = \sum_j \int id\, d(E_j)_{P_j x, P_j y}$

$= \sum_j \langle \cancel{P_j T x, y} \rangle \langle TP_j x, P_j y\rangle = \sum_j \langle TP_j x, y\rangle =$

$\underset{(3)}{=} \langle Tx, y\rangle$

(9) Uniqueness: Let $T = \int id\, dE$

$\Rightarrow I + T^*T = \int (1+|z|^2)\, dE$ (Thm 29 (c))

$\Rightarrow B = \int \frac{1}{1+|z|^2}\, dE$ (Thm 29 (d))

$C = \int \frac{z}{1+|z|^2}\, dE$ (Thm 29 (5))

Set $A_j = \{z \in \mathbb{C} ;\ \frac{1}{1+|z|^2} \in (\frac{1}{j+1}, \frac{1}{j}]$

$\Rightarrow P_j = \int \chi_{A_j}\, dE$ (using Lemma 37)

$TP_j = \int z \chi_{A_j}(z)\, dE$ (Th 29 (5))

Prop 32

$\Rightarrow$ E_j (the spectral measure of TP_j) is the image of E by $z \mapsto z\chi_{A_j}(z)$

So, $E_j(A) = E(\{z;\ z\chi_{A_j}(z) \in A\}) = \begin{cases} E(A \cap A_j), & 0 \notin A \\ E((A \cap A_j) \cup (\mathbb{C} \setminus A_j)) & \text{if } 0 \in A \end{cases}$

So, $E_j(A)P_j = E(A \cap A_j)$, hence $E(A) = \sum_j E_j(A)P_j$.

So, the definition from (7) is the unique possible

<u>Corollary X.41</u> T normal. Then T bdd $\Leftrightarrow$ $\sigma(T)$ bdd

<u>Proof</u>: The same as Corollary 37, just use Theorem 40

<u>Corollary X.42</u> $T = \int f\,dE$ $\Rightarrow$ the spectral measure of T is the image of E by f

$\ulcorner$ Proof: $F := f(E)$ in the sense of L 35

By L 35 $\int id\,dF = \int f\,dE = T$

So, by the uniqueness part of Thm 40, F is the spectral measure of T. $\lrcorner$