Theorem 7.45: T .. selfadjoint operator

$E :=$ the spectral measure of T. Since $\sigma(T) \subset \mathbb{R}$, we have $E(\mathbb{C} \setminus \mathbb{R}) = 0$

$E_\lambda := E((-\infty, \lambda])$, $\lambda \in \mathbb{R}$

(a) E_λ is an OG projection for $\lambda \in \mathbb{R}$
[by property (ii) of abstract spectral measures]

(b) $E_\lambda E_\mu = E_\mu E_\lambda = E_{\min\{\lambda, \mu\}}$, $\lambda, \mu \in \mathbb{R}$

[Use property (v): $E_\lambda E_\mu = E((-\infty, \lambda]) E((-\infty, \mu]) =$
$= E((-\infty, \lambda] \cap (-\infty, \mu]) =$
$= E((-\infty, \min\{\lambda, \mu\}])$]

(c) $E_\lambda x = \lim_{\mu \to \lambda+} E_\mu x$, $x \in H$, $\lambda \in \mathbb{R}$

[$\lim_{\mu \to \lambda+} E_\mu x = \lim_{\mu \to \lambda+} E((-\infty, \mu]) x$

$\|E_\lambda x - E_\mu x\| = \|E((\lambda, \mu]) x\| = \left\| \left(\int \chi_{(\lambda, \mu]} dE \right) x \right\|$

$= \left(\int |\chi_{(\lambda, \mu]}|^2 dE_{x,x} \right)^{1/2} = \left(E_{x,x}((\lambda, \mu]) \right)^{1/2} \xrightarrow{\mu \to \lambda+} \left(E_{x,x}(\emptyset) \right)^{1/2} = 0$]

(d) λ is not an eigenvalue of $T \Rightarrow \lim_{\mu \to \lambda-} E_\mu x = E_\lambda x$

[By the computation made in (c) we see

$\|E_\lambda x - E_\mu x\| \to \left(E_{x,x}(\{\lambda\}) \right)^{1/2} = 0$ (if λ is not an eigenvalue of T
(by Prop. 30)]

(e) λ is an eigenvalue of T $\Rightarrow$ $\lim\limits_{\mu\to\lambda-} E_\mu x =: P_\lambda x$ defines an OG projection s.t.
$E_\lambda - P_\lambda$ is also an OG projection and $R(E_\lambda - P_\lambda) = \ker(\lambda I - T)$

$\ulcorner$ It is enough to observe that $P_\lambda = E((-\infty,\lambda))$
$E_\lambda - P_\lambda = E(\{\lambda\})$
and use Prop. 30

That $\lim\limits_{\mu\to\lambda-} E_\mu x = E(\cancel{(}-\infty,\lambda))x$ can so computed similarly as (c) $\lrcorner$

(f) $\lim\limits_{\mu\to-\infty} E_\mu x = 0$, $\lim\limits_{\mu\to+\infty} E_\mu x = x$

$\ulcorner$ As in (c): $\|E_\mu x\| = \|E((-\infty,\mu])x\| = \left(E_{x,x}((-\infty,\mu])\right)^{1/2} \xrightarrow{\mu\to-\infty} 0$

$\|x - E_\mu x\| = \|E((\mu,+\infty))x\| = \left(E_{x,x}((\mu,+\infty))\right)^{1/2} \xrightarrow{\mu\to+\infty} 0$ $\lrcorner$

(g) $\lambda\in\mathbb{R}$: $\lambda\in\rho(T) \Leftrightarrow \mu\mapsto E_\mu$ is constant on a nbhd of λ

$\ulcorner$ $\Rightarrow$: $\lambda\in\rho(T) \Rightarrow \lambda\in \mathbb{R}\setminus\sigma(T)$, $E(\mathbb{R}\setminus\sigma(T)) = 0$.
$\sigma(T)$ closed $\Rightarrow \exists\ \delta>0$ $(\lambda-\delta,\lambda+\delta)\subset\rho(T)$,
Then $E((\lambda-\delta,\lambda+\delta)) = 0 \Rightarrow E_\mu$ is constant on $(\lambda-\delta,\lambda+\delta)$

$\Leftarrow$ Recall that $T = \int id\, dE$ and $\sigma(T) = \text{ess-rg}(id)$

If $\lambda\in\mathbb{R}$, $\delta>0$ s.t. E_μ is constant on $[\lambda-\mu,\lambda+\mu]$.
Then $E((-\infty,\lambda-\mu]) = E((-\infty,\lambda+\mu]) \Rightarrow E((\lambda-\mu,\lambda+\mu]) = 0$.
So, $\lambda\notin \text{ess-rang}(id) \Rightarrow \lambda\notin\sigma(T) \Rightarrow \lambda\in\rho(T)$ $\lrcorner$