Proof of Lemma XI.2

Let X be a vector space, $p_1, \dots, p_n$ seminorms on X,
$f \in X^{\#}$, $|f| \le \max(p_1, \dots, p_n)$

Consider X with the topology $\tau = \sup \mathcal{LC}(X)$, the strongest locally convex topology. Then $(X, \tau)^* = X^{\#}$ and all the seminorms are continuous.

Let $A_j := \{x \in X;\ p_j(x) \le 1\}$, $j = 1, \dots, n$

Then A_j is a closed, absolutely convex neighborhood of 0.

Thus A_j^0 is $\sigma(X^{\#}, X)$-compact (Banach–Alaoglu)
and $(A_j^0)_0 = A_j$ (bipolar theorem)

$$\text{Moreover, } f \in \Big(\bigcap_{j=1}^n A_j\Big)^0 = \Big(\bigcap_{j=1}^n (A_j^0)_0\Big)^0 =$$

$$= \Big(\Big(\bigcup_{j=1}^n A_j^0\Big)_0\Big)^0 \underset{\text{bipolar theorem}}{=} \overline{\operatorname{aco}\Big(\bigcup_{j=1}^n A_j^0\Big)}^{\sigma(X^{\#}, X)}$$

- A_j^0 are absolutely convex, hence $\bigcup_{j=1}^n A_j^0$ is balanced,

  therefore $\operatorname{aco}\big(\bigcup_{j=1}^n A_j^0\big) = \operatorname{co}\big(\bigcup_{j=1}^n A_j^0\big)$

- Further, A_j^0 are convex and $\sigma(X^{\#}, X)$-compact,

  thus $\operatorname{co}\big(\bigcup_{j=1}^n A_j^0\big)$ is also $\sigma(X^{\#}, X)$-compact

$$\Big[\ \operatorname{co}\Big(\bigcup_{j=1}^n A_j^0\Big) \underset{A_j^0 \text{ convex.}}{=} \Big\{\sum_{j=1}^n t_j g_j;\ g_j \in A_j^0,\ t_j \in [0,1],\ \sum_{j=1}^n t_j = 1\Big\}$$

Moreover, $K = \{(t_1, \dots, t_n) \in [0,1]^n ; \sum t_j = 1\}$ is compact, hence $U_1^0 \times \dots \times U_n^0 \times K$ is compact as well and the mapping $\Phi : U_1^0 \times \dots \times U_n^0 \times K \to X^\#$ defined by

$$\Phi(g_1, \dots, g_n, t_1, \dots, t_n) = \sum_{j=1}^n t_j g_j \text{ is continuous,}$$

and maps the compact space $U_1^0 \times \dots \times U_n^0 \times K$ onto $co\left(\bigcup_{j=1}^n A_j^0\right)$, so the last set is $\sigma(X^\#, X)$-compact, thus $\sigma(X^\#, X)$-closed.

It follows that $f \in co\left(\bigcup_{j=1}^n A_j^0\right)$, i.e., there are $f_j \in A_j^0$, $j = 1, \dots, n$, $t_1, \dots, t_n \in [0,1]$ with $\sum_{j=1}^n t_j = 1$ s.t. $f = \sum_{j=1}^n t_j f_j$. Since $f_j \in A_j^0 = \{g \in X^\#, |g| \le 1 \text{ on } A_j\}$

$$= \{g \in X^\# ; |p_j(x)| \le 1 \Rightarrow |g(x)| \le 1\},$$

we conclude that $|f_j| \le p_j$.

This completes the proof.

Proof of Proposition XI.3 . Let $\mathcal{F} \subset \mathcal{LC}(X)$

(1) $(X, \sup \mathcal{F})^* = \operatorname{span}_{\mathcal{T} \in \mathcal{F}} \left(\bigcup_{\mathcal{T} \in \mathcal{F}} (X, \mathcal{T})^* \right)$

$\supset$: $\mathcal{T} \in \mathcal{F} \Rightarrow \mathcal{T} \subset \sup \mathcal{F} \Rightarrow (X, \mathcal{T})^* \subset (X, \sup \mathcal{F})^*$

$\subset$: $f \in (X, \sup \mathcal{F})^* \Rightarrow$ (using description of $\sup \mathcal{F}$ and characterization of cts linear functionals)

$\exists\, \mathcal{T}_1, \dots, \mathcal{T}_n \in \mathcal{F}$, seminorms p_j cts on $\mathcal{T}_j$, $j = 1 \dots n$

s.t. $|f| \leq \max\{p_1, \dots, p_n\}$. By L2 we get

$f_1, \dots, f_n \in X^{\#}$, $|f_j| \leq p_j$, $f = \sum_{j=1}^{n} \varepsilon_j f_j$

Since $f_j \in (X, \mathcal{T}_j)^*$, we deduce $f \in \operatorname{span}_{\mathcal{T} \in \mathcal{F}} \left(\bigcup_{\mathcal{T} \in \mathcal{F}} (X, \mathcal{T})^* \right)$

(2) $(X, \inf \mathcal{F})^* = \bigcap_{\mathcal{T} \in \mathcal{F}} (X, \mathcal{T})^*$

[$f \in (X, \inf \mathcal{F})^* \Leftrightarrow |f|$ is an $(\inf \mathcal{F})$-cts seminorm

$\Leftrightarrow \forall \mathcal{T} \in \mathcal{F}$: $|f|$ is a $\mathcal{T}$-cts seminorm

$\Leftrightarrow \forall \mathcal{T} \in \mathcal{F}$: $f \in (X, \mathcal{T})^*$]