SELF-ADJOINT LAPLACE OPERATORS

Let $\Omega \subset \mathbb{R}^d$ be an open set.

① Denote $Z := W^{1,2}(\Omega) = \{ f \in L^2(\Omega) ; \forall j = 1 \dots d : \partial_j f \in L^2(\Omega) \}$

in $\mathcal{D}'(\Omega)$

Then Z is a Hilbert space, the inner product is defined by $\langle f, g \rangle_Z = \int_\Omega f \bar{g} + \sum_{j=1}^{d} \int_\Omega \partial_j f \, \partial_j \bar{g}$

Shortcut: $\nabla f = (\partial_1 f, \dots, \partial_d f)$ ... a vector function $\Omega \to \mathbb{C}$

Then $\langle f, g \rangle_Z = \int_\Omega f \bar{g} + \int_\Omega \langle \nabla f(x), \nabla g(x) \rangle_{\mathbb{C}^d} \, dx$

Let $Z_0 := \overline{\mathcal{D}(\Omega)}^{Z}$. Fix a closed subspace Y s.t. $Z_0 \subset Y \subset Z$

② Let $J : Y \to L^2(\Omega)$ be the "identity" mapping, i.e. $Jf = f$, $f \in Y$

Then • J is a bdd linear operator, $\|J\| \leq 1$

$$\Big[\|Jf\|^2_{L^2(\Omega)} = \langle Jf, Jf \rangle_{L^2(\Omega)} = \int_\Omega |f|^2 \leq$$

$$\leq \int_\Omega |f|^2 + \int_\Omega \langle \nabla f(x), \nabla f(x) \rangle \, dx = \|f\|^2_Y \Big]$$

• $R(J)$ is dense in $L^2(\Omega)$, as it contains $\mathcal{D}(\Omega)$

③ Let $J^* : L^2(\Omega) \to Y$ be the adjoint operator, i.e.

$\langle Jf, g \rangle_{L^2(\Omega)} = \langle f, J^* g \rangle_Y$ for $f \in Y$, $g \in L^2(\Omega)$.

Then $\|J^*\| = \|J\| \leq 1$, and J^* is one-to-one (as $R(J)$ is dense)

(4) The operator $JJ^* : L^2(\Omega) \to L^2(\Omega)$ is a bdd $(\|JJ^*\| \leq 1)$ self-adjoint $((JJ^*)^* = (J^*)^* J^* = JJ^*)$ operator.
Moreover, JJ^* is one-to-one:

• $JJ^* f = 0 \Rightarrow 0 = \langle JJ^* f, f \rangle = \langle J^* f, J^* f \rangle = \|J^* f\|^2$
$\Rightarrow J^* f = 0 \Rightarrow f = 0$ (as J^* is one-to-one).
Hence, $R(JJ^*)$ is dense and $T := (JJ^*)^{-1}$ is self-adjoint
(5) (Prop. XI.17 (e))
(5) $D(T) = R(JJ^*) = J(R(J^*))$. Since J is „identify", let us describe $R(J^*)$

Let $f \in Y$. Then $f \in R(J^*) \Leftrightarrow \exists g \in L^2(\Omega) : J^* g = f$

Further, $J^* g = f \Leftrightarrow \forall\ h \in Y : \langle J^* g, h \rangle_Y = \langle f, h \rangle_Y$

We have $\langle f, h \rangle_Y = \int_\Omega f \bar{h} + \int_\Omega \langle \nabla f, \nabla h \rangle$

$$\langle J^* g, h \rangle_Y = \langle g, Jh \rangle_Y = \int_\Omega g \bar{h}$$

So, $\langle f, h \rangle_Y = \langle J^* g, h \rangle_Y \Leftrightarrow \int_\Omega \langle \nabla f, \nabla h \rangle = \int_\Omega (g - f) \bar{h}$

Conclusion:

$$f \in R(J^*) \Leftrightarrow \exists g \in L^2(\Omega)\ \forall h \in Y : \int_\Omega \langle \nabla f, \nabla h \rangle = \int_\Omega (g - f) \bar{h}$$

(6) Necessary condition: $f \in D(T) \Rightarrow \Delta f$ (in $\mathcal{D}'(\Omega)$) belongs to $L^2(\Omega)$ and $Tf = f - \Delta f$

⌈ Apply ~~to~~ the characterization from (5) to $h \in \mathcal{D}(\Omega)$. Note, that $\mathcal{D}(\Omega) \subset Y$

So, $f \in R(J^*) \Rightarrow \exists g \in L^2(\Omega)\ \forall h \in \mathcal{D}(\Omega)\ \int_\Omega \langle Df, Dh \rangle = \int_\Omega (g-f)\overline{h}$

So, for $h \in \mathcal{D}(\Omega)$ we have

$$\int_\Omega (g-f)\overline{h} = \int_\Omega \langle Df, Dh \rangle = \sum_{j=1}^{d} \int_\Omega \partial_j f\, \overline{\partial_j h} =$$

$$= \sum_{j=1}^{d} \langle \partial_j f, \overline{\partial_j h} \rangle = - \sum_{j=1}^{d} \langle \partial_{jj} f, \overline{h} \rangle =$$

$$= \langle -\Delta f, \overline{h} \rangle$$ (where are applications of a distribution to a test function).

It follows $g - f = -\Delta f$ in $\mathcal{D}'(\Omega)$. Thus $\Delta f \in L^2(\Omega)$

and $g = f - \Delta f$

Since $f - \Delta f = g$ ~~x~~ and $f = J^* g$, we have

$Jf = JJ^* g \Rightarrow g = T(Jf)$

So, $T(Jf) = f - \Delta f$. This is the sought formula. ⌋

(7) Sufficient condition: $\mathcal{D}(\Omega) \subset D(T)$

Γ $f \in \mathcal{D}(\Omega) \Rightarrow g := f - \Delta f \in \mathcal{D}(\Omega) \subset L^2(\Omega)$

and for any $h \in Y$ we have

$$\int_\Omega (g-f)\bar{h} = \int_\Omega -\Delta f \bar{h} = -\sum_{j=1}^{d} \int_\Omega \bar{h} \cdot \partial_{jj} f =$$

$$= -\sum_{j=1}^{d} \langle \bar{h}, \partial_{jj} f \rangle = \sum_{j=1}^{d} \langle \partial_j \bar{h}, \partial_j f \rangle =$$

application of a distribution to a test function

$$= \sum_{j=1}^{d} \int_\Omega \partial_j \bar{h}\, \partial_j f = \int_\Omega \langle Df, Dh \rangle$$

$h \in Y \subset Z \Rightarrow \partial_j h \in L^2(\Omega)$.

So, $f \in R(J^*)$ by (5). $\lrcorner$

(8) Characterization of $D(T)$:

$f \in D(T) \Leftrightarrow f \in Y$ (more precisely, $J(Y)$) ; $\Delta f \in L^2(\Omega)$

& $\forall h \in Y$: $\int_\Omega (\langle Df, Dh \rangle + \Delta f \bar{h}) = 0$

$\Rightarrow$: $f \in D(T) \Rightarrow f \in J(Y)$ (by construction), $\Delta f \in L^2(\Omega)$ (by (6))

and $Tf = f - \Delta f$ (by (6)). By (5) we have

$\forall h \in Y$: $\int_\Omega \langle Df, Dh \rangle = \int_\Omega -\Delta f \bar{h}$

$\Leftarrow$: Let $g = f - \Delta f \in L^2(\Omega)$. Then by (5) we deduce $f \in D(T)$