

Proposition V.6

H -Hilbert space,

$P \in \mathcal{L}(H)$ a projection, i.e. $P^2 = P$

(iii) \Rightarrow (iii') P self-adjoint $\Rightarrow P$ normal
[trivial]

(iii') \Rightarrow (ii) P normal $\Rightarrow P$ orthogonal (i.e. $\ker P \perp \text{R}(P)$)

[by Prop. 5 (a)]

(ii) \Rightarrow (iv) P orthogonal $\Rightarrow \langle P_+, + \rangle = \|P_+\|^2, + \in H$

[$\langle P_+, + \rangle = \langle P_+, P_+ \rangle + \underbrace{\langle P_+, + - P_+ \rangle}_{\in \ker P} = \langle P_+, P_+ \rangle$]

(iv) \Rightarrow (v) $\forall + \in H \langle P_+, + \rangle = \|P_+\|^2 \Rightarrow \forall + \in H \langle P_+, + \rangle \geq 0$

[obvious]

(v) \Rightarrow (iii) $\forall + \in H \langle P_+, + \rangle \geq 0 \Rightarrow P^* = P$

[$\forall x \in H \langle P^* x, + \rangle = \langle +, P_+ \rangle = \overline{\langle P_+, + \rangle} = \langle P_+, + \rangle \Rightarrow P^* = P$]
↑
Prop. 4(c)

(iii) \Rightarrow (vi) $\forall x \in H : \langle P_+, + \rangle = \|P_+\|^2 \Rightarrow \|P\| \leq 1$

[$\forall + \in \mathcal{B}_H \Rightarrow \|P_+\|^2 = \langle P_+, + \rangle \leq \|P_+\| \cdot \|+\| \leq \|P\| \|+\|$]

So, $\|P\|^2 \leq \|P\|$. Hence $\|P\| \leq 1$

$(vi) \Rightarrow (c) \quad \|P\| \leq 1 \Rightarrow$ Orthogonal

• $(\text{Ker } P)^\perp \subset R(P)$

$\Gamma x \in (\text{Ker } P)^\perp \Rightarrow Tx = Px \in \text{Ker } P$, so

$\langle Px - P, Px - P \rangle = 0$, hence $\langle Px, Px \rangle = \langle P, P \rangle$

Thus $\|x\|^2 = \langle Px, Px \rangle \leq \|P\| \cdot \|x\| \leq \|x\|^2$

So, we have equalities. Thus $\|Px\| = \|x\|$

and Px is a multiple of x

$Px = \alpha x$

But $\langle Px, Px \rangle = \langle \alpha x, \alpha x \rangle =$

$\alpha^2 \langle x, x \rangle$, hence

necessarily $\alpha = 1$ (or $\alpha = 0$, but

this is a trivial case)

Hence $Px = x$, so $x \in R(P)$.

• $R(P) \subset (\text{Ker } P)^\perp$

$\Gamma x \in R(P) \Rightarrow x = x_1 + x_2$, $x_1 \in \text{Ker } P$, $x_2 \in (\text{Ker } P)^\perp$

By above $x_2 \in R(P)$.

Thus $x = Px = Px_1 + Px_2 = Px_2 = x_2 \in (\text{Ker } P)^\perp$

Moreover, if P, Q are OG projectors, then $R(P) \perp R(Q) \Leftrightarrow PQ = 0$

$\Gamma R(P) \perp R(Q) \Leftrightarrow R(Q) \subset R(P)^\perp = \text{Ker } P \Leftrightarrow PQ = 0 \Gamma$