Proof of Proposition VI.24

Let E be an abstract spectral measure in H,
$\mathcal{A}$ the domain σ-algebra
for $f: \mathbb{C} \to \mathbb{C}$ $\mathcal{A}$-measurable set $\Phi(f) = \int f \, dE$

(1) For $x \in H$ set $H_x := \{\Phi(f)x \, ; \, f \in C_b(\mathbb{C})\}$

Then • $H_x \subset\subset H$ (as Φ is linear), not necess. closed
• $f \in C_b(\mathbb{C}) \Rightarrow \|\Phi(f)x\| = \|f\|_{L^2(E_{x,x})}$ (Th 8(d))

For $f \in L^2(E_{x,x})$ define $U_x(f) = \Phi(f)x$

Then U_x is a linear isometry of $L^2(E_{x,x})$ into H (by Th 11)

Moreover, the range of U_x is $\overline{H_x}$ (cts functions are dense in $L^2(E_{x,x})$)

(2) If $y \perp H_x$, then $H_y \perp H_x$ (hence $\overline{H_y} \perp \overline{H_x}$)

[Let $y \perp H_x$. Fix $f, g \in C_b(\mathbb{C})$. Then

$$\langle \Phi(g)y, \Phi(f)x \rangle = \langle y, \Phi(g)^* \Phi(f)x \rangle \overset{\text{Th 8(a)}}{=}$$

$$= \langle y, \underbrace{\Phi(\overline{g} \cdot f)x}_{\in H_x} \rangle = 0$$]

(3) Let $\Gamma \subset S_H$ be maximal s.t. $H_x \perp H_y$, $x, y \in \Gamma$, $x \neq y$
(It exists by Zorn's Lemma)

Then $\mathrm{span}\left(\bigcup_{x \in \Gamma} H_x\right)$ is dense in H [otherwise $\exists$ $z \in S_H$
$z \perp \bigcup_{x \in \Gamma} H_x \Rightarrow z$
can be added to Γ]

(4) Let $\Omega = \Gamma \times \mathbb{C}$ Define a σ-algebra $\tilde{\mathcal{A}}$ on Ω by

$$\tilde{\mathcal{A}} = \{ A \subset \Omega ;\ \forall x \in \Gamma : \{ \lambda \in \mathbb{C} ;\ (x, \lambda) \in A \} \in \mathcal{A}$$

Then $\tilde{\mathcal{A}}$ is clearly a σ-algebra. For $A \in \tilde{\mathcal{A}}$ set

$$\mu(A) = \sum_{x \in \Gamma} E_{x,x}\{ \lambda \in \mathbb{C},\ (x, \lambda) \in A \}$$

Then μ is a nonnegative σ-additive measure on Ω

(5) Define $U : L^2(\mu) \to H$ by

$$U(g) = \sum_{x \in \Gamma} \Phi(\lambda \mapsto g(x, \lambda))x, \quad g \in L^2(\mu)$$

Then U is a linear isometry of $L^2(\mu)$ onto H:

- Fix $g \in L^2(\mu)$. For $x \in \Gamma$ set $g_x(\lambda) = g(x, \lambda), \lambda \in \mathbb{C}$
  Then $g_x \in L^2(E_{x,x})$. Moreover,

by (1) $\Phi(g_x)x \in \overline{H_x}$, $\| \Phi(g_x)x \| = \|g_x\|_{L^2(E_{x,x})}$

- Further $\|g\|_{L^2(\mu)} = \left(\sum \|g_x\|^2_{L_2(E_{x,x})} \right)^{1/2}$

Since $\overline{H_x} \perp \overline{H_y}$ for $y \neq x$, we conclude

$$\| U(g) \| = \|g\|_{L^2(\mu)},$$

Since U is clearly linear, we deduce that U is a linear isometry

- U is onto: $z \in H \Rightarrow z = \sum_{x \in \Gamma} z_x$ for some $z_x \in \overline{H_x}, x \in \Gamma$

Since U_x is onto ... $\exists f_x \in L^2(E_{x,x})$, $U_x(f_x) = z_x$, $x \in \Gamma$

$f(x, \lambda) = f_x(\lambda)$, $(x, \lambda) \in \Omega \Rightarrow f \in L^2(\mu)$, $U(f) = z$

(6) Let $f: \mathbb{C} \to \mathbb{C}$ be $\mathcal{A}$-measurable

Set $\tilde{f}(x,\lambda) = f(\lambda)$, $(x,\lambda) \in \Omega \Rightarrow \tilde{f}$ is $\tilde{\mathcal{A}}$-measurable

We claim that $\Phi(f) = U M_{\tilde{f}} U^*$, i.e. $M_{\tilde{f}} = U^* \Phi(f) U$

Recall $M_{\tilde{f}}(g) = \tilde{f} \cdot g$, $g \in D(M_{\tilde{f}}) = \{g, g \in L^2(\mu);\ \tilde{f} \cdot g \in L^2(\mu)\}$

- $g \in D(M_{\tilde{f}}) \Leftrightarrow g \cdot \tilde{f} \in L^2(\mu) \Leftrightarrow \sum_{x \in \Gamma} \int |g(x,\lambda) \cdot f(\lambda)|^2 < \infty$

- $D(U^* \Phi(f) U) = \{g \in D(U);\ Ug \in D(\Phi(f))\}$, $D(U) = L^2(\mu)$

So: $g \in D(U^* \Phi(f) U) \Leftrightarrow Ug \in D(\Phi(f)) \Leftrightarrow f \in L^2(E_{U(g), U(g)}) \Leftrightarrow$

$\ulcorner$ $E_{U(g),U(g)}(A) = \langle E(A) U(g), U(g) \rangle = \langle E(A) U(g), \sum_{x \in \Gamma} U_x g_x \rangle$

$= \sum_{x \in \Gamma} \langle E(A) U(g), U_x g_x \rangle = \sum_{x,y \in \Gamma} \langle E(A) U_y g_y, U_x g_x \rangle =$

$$= \sum_{x,y \in \Gamma} \langle \Phi(\chi_A) \Phi(g_y) y, \Phi(g_x) x \rangle \overset{\text{Thm 12 (b)}}{=}$$

$$= \sum_{x,y \in \Gamma} \langle \Phi(\chi_A g_y) y, \Phi(g_x) x \rangle \overset{H_x \perp H_y \text{ for } x \neq y}{=} \sum_{x \in \Gamma} \langle \Phi(\chi_A g_x) x, \Phi(g_x) x \rangle$$

$$\overset{\text{Thm 12 (c), (b)}}{=} \sum_{x \in \Gamma} \langle \Phi(\chi_A g_x \overline{g_x}) x, x \rangle$$

$$= \sum_{x \in \Gamma} \int_A |g_x|^2 \, dE_{x,x} = \int_{\Gamma \times A} |g|^2 \, d\mu \quad \lrcorner$$

$\Leftrightarrow \int |f|^2 \, d E_{U(g)U(g)} < \infty \Leftrightarrow \int |\tilde{f}|^2 \cdot |g|^2 \, d\mu < \infty$

$\Leftrightarrow \tilde{f} \cdot g \in L^2(\mu) \lrcorner$

So, $D(M_{\tilde{f}}) = D(U^* \Phi(f) U)$

Moreover, if $g, h \in$ common domain, then

$$\langle U^* \Phi(f) U g, h \rangle = \langle \Phi(f) U g, U h \rangle =$$

$$= \sum_{x,y \in \Gamma} \langle \Phi(f) \Phi(g_x) x, \Phi(h_y) y \rangle \overset{\text{Th } 8(3)}{=} \sum_{x,y \in \Gamma} \langle \Phi(f g_x) x, \Phi(h_y) y \rangle$$

$H_x \perp H_y$ for $x \neq y$

$$= \sum_{x \in \Gamma} \langle \Phi(f g_x) x, \Phi(h_x) x \rangle \overset{\varnothing}{=} \sum_{x \in \Gamma} \cancel{e_{x, y, \mu} \Phi(f g_x)^* \Phi(h_x) x}$$

$$= \sum_{x \in \Gamma} \lim_{n \to \infty} \langle \Phi_0 ((f g_x)_n) x, \Phi((h_x)_n) x \rangle =$$

$$= \sum_{x \in \Gamma} \lim_{n \to \infty} \langle \Phi_0 ((\bar{h}_x f g_x)_n x, x \rangle = \sum_{x \in \Gamma} \cancel{\langle \Phi_0 f}$$

$$= \sum_{x \in \Gamma} \lim_{n \to \infty} \int (\bar{h}_x f g_x)_n \, dE_{x,x} = \sum_{x \in \Gamma} \int \bar{h}_x f g_x \, dE_{x,x}$$

$$= \int \bar{h} \tilde{f} g \, d\mu = \langle M_{\tilde{f}} g, h \rangle$$

Proof of Theorem VI.25

Let T be a normal operator on H.

Let $E = E_T$ be its spectral measure

Let μ and $U: L^2(\mu) \to H$ be as in Prop. 43

Then $T = U M_{\widetilde{id}} U^*$, using the notation from Prop. 43,

as $T = \int id \, dE$

Moreover:

- T self-adjoint $\Rightarrow$ id is essentially real-valued, so $\widetilde{id}$ is also essentially real-valued
- T bdd $\Rightarrow$ id is essentially bdd $\Rightarrow$ $\widetilde{id}$ is essentially bdd
- H separable $\Rightarrow$ μ can be chosen to be σ-finite, as the set P from the proof of ~~Thm~~ Prop. 24 is countable.