Theorem VI.26 T .. selfadjoint operator

E := the spectral measure of T. Since $\sigma(T) \subset \mathbb{R}$, we have $E(\mathbb{C} \setminus \mathbb{R}) = 0$

$E_\lambda := E((-\infty, \lambda])$, $\lambda \in \mathbb{R}$

(a) E_λ is an OG projection for $\lambda \in \mathbb{R}$
[by property (ii) of abstract spectral measures]

(b) $E_\lambda E_\mu = E_\mu E_\lambda = E_{\min\{\lambda, \mu\}}$, $\lambda, \mu \in \mathbb{R}$

[Use property (v): $E_\lambda E_\mu = E((-\infty,\lambda]) E((-\infty,\mu]) =$
$= E((-\infty,\lambda] \cap (-\infty,\mu]) =$
$= E((-\infty, \min\{\lambda,\mu\}])$]

(c) $E_\lambda x = \lim_{\mu \to \lambda+} E_\mu x$, $x \in H$, $\lambda \in \mathbb{R}$

[$\lim_{\mu \to \lambda+} E_\mu x = \lim_{\mu \to \lambda+} E((-\infty,\mu]) x$

$\|E_\lambda x - E_\mu x\| = \| E((\lambda,\mu])x\| = \| (\int \chi_{(\lambda,\mu]} dE) x \|$

$= \left(\int |\chi_{(\lambda,\mu]}|^2 dE_{x,x} \right)^{1/2} = \left(E_{x,x}((\lambda,\mu]) \right)^{1/2} \xrightarrow{\mu \to \lambda+} (E_{x,x}(\emptyset))^{1/2} = 0$]

(d) λ is not an eigenvalue of $T \Rightarrow \lim_{\mu \to \lambda -} E_\mu x = E_\lambda x$

[By the computation in (c) we see

$\|E_\lambda x - E_\mu x\| \to (E_{x,x}(\{\lambda\}))^{1/2} = 0$ (if λ is not an eigenvalue of T
(by Prop. 13)]

(e) λ is an eigenvalue of $T \Rightarrow \lim_{\mu \to \lambda-} E_\mu x =: P_\lambda x$ defines an OG projection s.t.

$E_\lambda - P_\lambda$ is also an OG projection and $R(E_\lambda - P_\lambda) = \ker(\lambda I - T)$

$\ulcorner$ It is enough to observe that $P_\lambda = E((-\infty, \lambda))$

$E_\lambda - P_\lambda = E(\{\lambda\})$

and use Prop. 13

That $\lim_{\mu \to \lambda-} E_\mu x = E(\cancel{(}(-\infty, \lambda))x$ can be computed similarly as (c) $\lrcorner$

(f) $\lim_{\mu \to -\infty} E_\mu x = 0$, $\lim_{\mu \to +\infty} E_\mu x = x$

$\ulcorner$ As in (c): $\|E_\mu x\| = \| E((-\infty, \mu])x\| = \left(E_{x,x}((-\infty, \mu])\right)^{1/2} \xrightarrow{\mu \to -\infty} 0$

$\|x - E_\mu x\| = \| E((\mu, +\infty))x\| = \left(E_{x,x}((\mu, +\infty))\right)^{1/2} \xrightarrow{\mu \to +\infty} 0$ $\lrcorner$

(g) $\lambda \in \mathbb{R}$: $\lambda \in \rho(T) \Leftrightarrow \mu \mapsto E_\mu$ is constant on a nbhd of λ

$\ulcorner \Rightarrow$: $\lambda \in \rho(T) \Rightarrow \lambda \in \mathbb{R} \setminus \sigma(T)$, $E(\mathbb{R} \setminus \sigma(T)) = 0$.

$\sigma(T)$ closed $\Rightarrow \exists\, \delta > 0$ $(\lambda - \delta, \lambda + \delta) \subset \rho(T)$,

The $E((\lambda - \delta, \lambda + \delta)) = 0 \Rightarrow E_\mu$ is constant on $(\lambda - \delta, \lambda + \delta)$

$\Leftarrow$ Recall that $T = \int id \, dE$ and $\sigma(T) = \text{ess-rg}(id)$

If $\lambda \in \mathbb{R}$, $\delta > 0$ s.t. E_μ is constant on $[\lambda - \mu, \lambda + \mu]$.

Then $E((-\infty, \lambda - \mu]) = E((-\infty, \lambda + \mu]) \Rightarrow E((\lambda - \mu, \lambda + \mu]) = 0$.

So, $\lambda \notin$ ess-rang$(id) \Rightarrow \lambda \notin \sigma(T) \Rightarrow \lambda \in \rho(T)$ $\lrcorner$