Theorem VI.27 Let H be a real Hilbert space

$$H_C = H + iH = \{x + iy \,;\, x, y \in H\}$$

$$\langle x+iy, u+iv \rangle := \langle x, u \rangle + \langle y, v \rangle + i\left(\langle y, u \rangle - \langle x, v \rangle\right)$$

$x+iy,\ u+iv \in H_C$

It is an inner product and H_C is a complex Hilbert space

Let T be an operator on H (with domain $D(T)$)

Define $T_C(x+iy) := T(x) + iT(y)$, $x+iy \in D(T_C) = \{x+iy,\ x, y \in D(T)\}$

Then:
1. T_C is a linear operator, $D(T_C)$ is a linear subspace
2. $D(T)$ dense $\Rightarrow D(T_C)$ dense in H_C
3. Suppose T densely defined $\Rightarrow (T_C)^* = (T^*)_C$

$\supset$ $u, v \in D(T^*)$, $x+iy \in D(T_C)$

$$\langle T_C(x+iy), u+iv \rangle = \langle Tx + iTy, u+iv \rangle =$$
$$= \langle Tx, u \rangle + \langle Ty, v \rangle + i\left(\langle Ty, u \rangle - \langle Tx, v \rangle\right) =$$
$$= \langle x, T^*u \rangle + \langle y, T^*v \rangle + i\left(\langle y, T^*u \rangle - \langle x, T^*u \rangle\right) =$$
$$= \langle x+iy, T^*u + iT^*v \rangle = \langle x+iy, (T^*)_C(u+iv) \rangle$$

$\subset$: $u+iv \in D((T_C)^*)$

$$\Rightarrow \exists\ w+iz \in H_C \text{ s.t. } \forall x+iy \in D(T_C):$$
$$\langle T_C(x+iy), u+iv \rangle = \langle x+iy, w+iz \rangle$$

So, $\langle Tx, u \rangle + \langle Ty, v \rangle = \langle x, w \rangle + \langle y, z \rangle$, $x, y \in D(T)$

(take the real part). apply for $y = 0 \Rightarrow u \in D(T^*)$, $T^*u = w$

apply for $x = 0 \Rightarrow v \in D(T^*)$, $T^*v = z$.

4. T self-adjoint $\Rightarrow T_C$ self-adjoint

(clear from 3)

(5) T self-adjoint, $\lambda \in \mathbb{R}$ $\Rightarrow$ ($\lambda I - T$ invertible $\Leftrightarrow$ $(\lambda I - T_C)$ invertible)

- $(\lambda I - T)x = 0 \Rightarrow (\lambda I - T_C)(x + i0) = 0$
- $(\lambda I - T_C)(x + iy) = 0$

$\Rightarrow (\lambda I - T)x + i(\lambda I - T)y = 0 \Rightarrow (\lambda I - T)x = 0$
$\& \; (\lambda I - T)y = 0$

So, $\lambda I - T$ is one-to-one $\Leftrightarrow$ $(\lambda I - T_C)$ is one-to-one

- $(\lambda I - T_C)(x + iy) = (\lambda I - T)x + i(\lambda I - T)y$

So, $(\lambda I - T_C)$ is onto $\Leftrightarrow$ $\lambda I - T$ is onto

(6) Suppose T is bdd and selfadjoint. Then $\forall f \in \mathcal{C}(\sigma(T_C))$ real-valued : $\tilde{f}(T_C) H \subset H$

[It holds for $f(t) = t^n$, $n \in \mathbb{N} \cup \{0\}$, by Stone-Weierstrass then it holds for $f \in \mathcal{C}(\sigma(T_C))$]

(7) The same holds for f bdd $\mathcal{B}_{T_C}$-measurable, real-valued

[Suppose $\tilde{f}(T_C)(H) \not\subset H$, so $\exists x \in H$

s.t. $\tilde{f}(T_C)(x) = u + iv$, $u, v \in H$, $v \neq 0$

$\Rightarrow \langle \tilde{f}(T_C)x, v \rangle = \langle u, v \rangle + i \langle v, v \rangle \in \mathbb{C} \setminus \mathbb{R}$

$\exists (g_n)$ cts, $g_n \to f$ $|E_{x,v}|$-a.e., $\|g_n\|_\infty \leq \|f\|_\infty$

Then $\langle \tilde{g_n}(T_C)x, v \rangle \longrightarrow \langle \tilde{f}(T_C)x, v \rangle \in \mathbb{C} \setminus \mathbb{R}$

$\in \mathbb{R}$; ^ Lebesgue dom. conv. thm

a contradiction]

(8) So, if T is bdd, self-adjoint, ~~then~~ and E is the spectral measure of T_C, then $E(A) H \subset H$ for $A \in \mathcal{A}^{\mathbb{C}}_{\mathbb{C}}$. Therefore $E_R(A) := E(A)|_H$ defines a „real spectral measure"

and $T = \int id \, dE_R$, since $\langle Tx, y \rangle = \int id \, dE_{x,y} = \int id \, d(E_R)_{x,y}$

$(E_R)_{x,y} = E_{x,y}$ for $x, y \in H$

(9) T unbdd self-adjoint $\Rightarrow$ T_C^2 is selfadjoint, $T_C^2(H) \subset H$

$I + T_C^2$ also self-adjoint, maps H into H, $D(T_C^2)$ onto H_C

so, it maps $D(T_C^2) \cap H$ onto H

(and L19)

$B := (I + T_C^2)^{-1} \in L(H_C)$, it is self-adjoint and $B(H) \subset H$

$C := T_C B = T_C(I + T_C^2)^{-1} \in L(H_C)$, self-adjoint, $C(H) \subset H$

$P_j := \tilde{\chi}_{(\frac{1}{j+1}, \frac{1}{j}]}(B) \Rightarrow P_j(H) \subset H$ by (8)

Then TP_j is a bdd self-adjoint operator (see the proof of Th 21)

So, E_j, the spectral measure of TP_j satisfies $E_j(A)H \subset H$, $A \in \mathcal{A}_j$

Since $E = \sum E_j$ is the spectral measure of T, we see

$E(A)H \subset H$, $A \in \mathcal{A}$

(10) HENCE: $E_R(A) := E(A)|_H$ defines a „real spectral measure" in H and $T = \int id \, dE_R$,

since $\forall y \in D(T) \Rightarrow$

$\langle (\int id \, dE_R) x, y \rangle = \int id \, d(E_R)_{x,y} = \int id \, dE_{x,y} =$

$= \langle T_C x, y \rangle = \langle Tx, y \rangle$

$\uparrow$

$((E_R)_{x,y} = E_{x,y}, \; x, y \in H)$