

Proof of Lemma VII.2.

Let X be a vector space, P_1, \dots, P_n seminorms on X ,
 $f \in X^\#$, $|f| \leq \max\{P_1, \dots, P_n\}$

Consider X with the topology $\sigma = \sup \mathcal{L}(X)$, the strongest
locally convex topology. Then $(X, \sigma)^\# = X^\#$ and all the
seminorms are continuous.

Let $A_j := \{x \in X; P_j(x) \leq 1\}$, $j=1, \dots, n$

Then A_j is a closed, absolutely convex neighborhood of 0 .

The A_j^0 is $\sigma(X^\#, X)$ -compact (Banach-Alaoglu)
and $(A_j^0)_0 = A_j$ (bipolar theorem)

$$\begin{aligned} \text{Moreover, } f \in \left(\bigcap_{j=1}^n A_j \right)^0 &= \left(\bigcap_{j=1}^n (A_j^0)_0 \right)^0 = \\ &= \left(\left(\bigcup_{j=1}^n A_j^0 \right)_0 \right)^0 = \overline{\text{aco} \left(\bigcup_{j=1}^n A_j^0 \right)}_{\sigma(X^\#, X)} \end{aligned}$$

↑
bipolar theorem

• A_j^0 are absolutely convex, hence $\bigcup_{j=1}^n A_j^0$ is balanced,
therefore $\text{aco} \left(\bigcup_{j=1}^n A_j^0 \right) = \text{co} \left(\bigcup_{j=1}^n A_j^0 \right)$

• Further, A_j^0 are convex and $\sigma(X^\#, X)$ -compact,
thus $\text{co} \left(\bigcup_{j=1}^n A_j^0 \right)$ is also $\sigma(X^\#, X)$ -compact

$$\uparrow \text{co} \left(\bigcup_{j=1}^n A_j^0 \right) = \left\{ \sum_{j=1}^n t_j g_j; g_j \in A_j^0, t_j \in [0, 1], \sum_{j=1}^n t_j = 1 \right\}$$

A_j^0 convex.

Moreover, $K = \{ (t_1, \dots, t_n) \in [0, 1]^n; \sum t_j = 1 \}$ is compact,

hence $A_1^0 \times \dots \times A_n^0 \times K$ is compact as well and

the mapping $\Phi: A_1^0 \times \dots \times A_n^0 \times K \rightarrow X^\#$ defined by

$$\Phi(g_1, \dots, g_n, t_1, \dots, t_n) = \sum_{j=1}^n t_j g_j$$
 is continuous,

and maps the compact space $A_1^0 \times \dots \times A_n^0 \times K$ into

$\text{co}(\bigcup_{j=1}^n A_j^0)$, so the last set is $\sigma(X^\#, X)$ -compact,

thus $\sigma(X^\#, X)$ -closed.

It follows that $f \in \text{co}(\bigcup_{j=1}^n A_j^0)$, i.e., there are

$$f_j \in A_j^0, j=1, \dots, n, \quad t_1, \dots, t_n \in [0, 1] \quad \text{with} \quad \sum_{j=1}^n t_j = 1.$$

$$s \in \text{co} \quad f = \sum_{j=1}^n s_j f_j. \quad \text{Since} \quad f_j \in A_j^0 = \{g \in X^\#, |g| \leq 1 \text{ on } A_j\}$$

$$= \{g \in X^\#; |P_j \cdot g| \leq 1 \Rightarrow |g(x_j)| \leq 1\},$$

we conclude that $|f \cdot s| \leq P_0$.

This completes the proof.

Proof of Proposition VII.3 . Let $\mathcal{F} \subset \mathcal{L}(X)$

$$(1) (x, \sup \mathcal{F})^* = \text{span}_{T \in \mathcal{F}} \left(\bigcup_{T \in \mathcal{F}} (x, T)^* \right)$$

$$\supset: T \in \mathcal{F} \Rightarrow T \subset \sup \mathcal{F} \Rightarrow (x, T)^* \subset (x, \sup \mathcal{F})^*$$

$$\subset: f \in (x, \sup \mathcal{F})^* \Rightarrow \text{(using description of } \sup \mathcal{F} \text{ and characterization of cts linear functionals)}$$

$$\exists T_1, \dots, T_n \in \mathcal{F}, \text{ seminorms } p_j, \text{ cts in } T_j, j=1, \dots, n$$

$$\text{s.t. } |f| \leq \max \{p_1, \dots, p_n\}. \text{ By L2 we get}$$

$$f_1, \dots, f_n \in X^*, |f_j| \leq p_j, f = \sum_{j=1}^n \epsilon_j f_j$$

$$\text{Since } f_j \in (x, T_j)^*, \text{ we deduce } f \in \text{span}_{T \in \mathcal{F}} \left(\bigcup_{T \in \mathcal{F}} (x, T)^* \right)$$

$$(2) (x, \inf \mathcal{F})^* = \bigcap_{T \in \mathcal{F}} (x, T)^*$$

$$\left\{ \begin{array}{l} f \in (x, \inf \mathcal{F})^* \Leftrightarrow |f| \text{ is an } (\inf \mathcal{F})\text{-cts} \\ \text{seminorm} \end{array} \right.$$

$$\Leftrightarrow \forall T \in \mathcal{F} : |f| \text{ is a } T\text{-cts seminorm}$$

$$\Leftrightarrow \forall T \in \mathcal{F} : f \in (x, T)^*$$