Proof of Lemma VII.4

$(X, \mathcal{T})$ ... LCS. Then $X^* \subset X^\#$

(a) The topology $\sigma(X^\#, X)$ is Hausdorff

⌈clear, as X separates points of $X^\#$
— $f \in X^\#$, $f \neq 0 \Rightarrow \exists x \in X$ $f(x) \neq 0$ ⌋

The topology $\sigma(X^*, X)$ on X^* is the subspace topology generated by $\sigma(X^\#, X)$

⌈clear from definitions⌋

(b) $(X, \mathcal{T})$ Hausdorff $\Rightarrow X^*$ is $\sigma(X^\#, X)$-dense in $X^\#$

⌈$(X, \mathcal{T})$ Hausdorff $\Rightarrow X^*$ separates points of X
(a consequence of the H-B theorem)

i.e. $(X^*)_\perp = \{0\}$. Thus $((X^*)_\perp)^\perp = X^\#$.

By the bipolar theorem we see that $X^\# = \overline{X^*}^{\sigma(X^\#, X)}$ ⌋

(c) $A \subset X^*$. Then A is $\sigma(X^*, X)$-relatively compact in X^*

$\Longleftrightarrow$
- A is $\sigma(X^*, X)$-bdd
- $\overline{A}^{\sigma(X^\#, X)} \subset X^*$

⌈$\Rightarrow$ A rel. compact $\Rightarrow$ $\overline{A}^{\sigma(X^*, X)}$ is $\sigma(X^*, X)$-compact

Thus $\forall x \in X$ $f \mapsto f(x)$ is bdd on A, i.e.,
A is $\sigma(X^*, X)$ bdd

• Moreover, $\overline{A}^{\sigma(X^*,X)}$ is $\sigma(X^*,X)$-compact, thus $\sigma(X^\#,X)$-compact, thus $\sigma(X^\#,X)$ closed (as the topology is Hausdorff)

It follows $\overline{A}^{\sigma(X^\#,X)} = \overline{A}^{\sigma(X^*,X)} \subset X$

$\Leftarrow$ Define $q_A(x) = \sup\{|f(x)|;\ f \in A\}$, $x \in X$

As A is $\sigma(X^*,X)$-bdd, it is a well-defined seminorm. So, it is $\sup \mathcal{L}C(X)$-cts.

• Thus $A_0 = \{x \in X;\ q_A(x) \le 1\}$ is a $\sup \mathcal{L}C(X)$-nbhd of 0. Therefore

$$\overline{\operatorname{aco} A}^{\sigma(X^\#,X)} = (A_0)^0 \text{ is } \sigma(X^\#,X)\text{-compact}$$

(Banach–Alaoglu)

↑ bipolar theorem

Thus also $\overline{A}^{\sigma(X^\#,X)}$ is $\sigma(X^\#,X)$-compact, therefore $\sigma(X^*,X)$-compact

<u>Lemma VII.6</u>: X vector space $A \subset X^\#$ $\sigma(X^\#,X)$ bdd, $f \in X^\#$. Then

$$|f| \le q_A \iff f \in \overline{\operatorname{aco} A}^{\sigma(X^\#,X)}$$

<u>Proof:</u>

Observe that $|f| \le q_A$ means that $f \in (A_0)^0$

and use the bipolar theorem