Proof of Theorem VII.6. (Mackey–Arens)

Let X be a vector space and $M \subset\subset X^{\#}$

Then $\mu(X,M)$ = the topology of uniform convergence on absolutely convex $\sigma(M,X)$-compact subsets of M

Let $\mathcal{T}$ denote the topology on the right-hand side

① Suppose $f \in (X,\mathcal{T})^*$. Then there are $A_1,\dots,A_n \subset M$ absolutely convex $\sigma(M,X)$-compact sets s.t. $|f| \le \max\{q_{A_1},\dots,q_{A_n}\}$

Let $A = \operatorname{co}\left(\bigcup_{j=1}^{n} A_j\right)$. Then A is absolutely convex and $\sigma(M,X)$-compact (see the proof of Lemma 2)

and $|f| \le q_A$. By Lemma 5 we deduce $f \in A \subset M$.

Thus $(X,\mathcal{T})^* \subset M$.

② Let p be a $\mu(X,M)$-cts seminorm on X.

Then $A = \{x \in X;\ p(x) \le 1\}$ is a $\mu(X,M)$-nbhd of 0 in X,

thus A^0 is a $\sigma(M,X)$-compact absolutely convex subset of M (Banach–Alaoglu, note that $(X,\mu(X,M))^* = M$)

Observe that $p = q_{A^0}$

[$q_{A^0}(x) \le 1 \Leftrightarrow x \in (A^0)_0 \overset{A \text{ absolutely convex, closed hence weakly closed}}{=} A \Leftrightarrow p(x) \le 1$, as p is the Minkowski functional of A]

Thus p is $\mathcal{T}$-cts. It follows $\mu(X,M) \subset \mathcal{T}$

③ By ② we have $\mu(X,M) \subset \mathcal{T}$, hence $\underbrace{(X,\mu(X,M))^*}_{=M} \subset (X,\mathcal{T})^*$

Combining with ① we see that $(X,\mathcal{T})^* = M$, hence $\mathcal{T} \subset \mu(X,M)$.

Together with ② this yields $\mathcal{T} = \mu(X,M)$.

Proposition VII.7 $(X,\mathcal{T})$ metrizable LCS

(a) $(X^*, \sigma(X^*,X))$ σ-compact

Γ $(U_n)_{n\in\mathbb{N}}$ a ctbl base of neighborhoods of 0. then U_n^0 is compact in $\sigma(X^*,X)$ by Banach-Alaoglu and

$X^* = \bigcup_{m,n} m\cdot U_n^0$: $f\in X^* \Rightarrow \exists n$ f is bdd on U_n

$\Rightarrow \exists m$ $|f|\le m$ on U_n ⅃

(b) $\mu(X,X^*) = \mathcal{T}$

Γ ⊃ clear

⊂ Let (U_n) be a ctbl base of nbds of 0 of $\mathcal{T}$ s.t. $U_1 \supset U_2 \supset U_3 \supset \dots$ (U_n can be absolutely convex)

Let V be an absolutely convex $\mu(X,X^*)$-nbhd of 0.
We shall prove that $\exists n$ s.t. $\frac{1}{n} U_n \subset V$.
If not, find $x_n \in \frac{1}{n}U_n \setminus V$, $n\in\mathbb{N}$.

Then $nx_n \to 0$ in $\mathcal{T}$, so $A=\{nx_n, n\in\mathbb{N}\}\cup\{0\}$ is $\mathcal{T}$-compact and hence $\mathcal{T}$-bdd.
Since $\mathcal{T}$-bdd and $\mu(X,X^*)$-bdd sets coincide, A is $\mu(X,X^*)$-bdd. It follows that $\exists r>0$ $A\subset rV$

Fix $n\in\mathbb{N}$, $n>r$. Then $A\subset nV$, hence $nx_n\in nV$, so $x_n\in V$, a contradiction.