

Proposition VII.11 Let X be a normed space. Then the bw^* -topology on X^* coincides with the topology of uniform convergence on sequences in X which converge to zero (in norm).

Proof (1) $U \subset X^*$ bw^* -open, $x^* \in X^* \Rightarrow x^* + U$ bw^* -open

$\Gamma \cdot U \subset X^*$ bw^* -open $\Rightarrow \forall A \subset X^*$ s.t. $A \cap U$ is (rel.) w^* -open in A .

$\Uparrow A$ s.t. $\exists r > 0 : A \cap rB_{X^*}$

$U \cap rB_{X^*}$ is w^* -open in rB_{X^*} , hence (s.t. θ depends of the $\text{scal} \neq 0$ topology) $U \cap A$ is w^* -open in A \downarrow

$\bullet U \subset X^*$ bw^* -open, $x^* \in X^*$, $r > 0 \Rightarrow$

$$(x^* + U) \cap rB_{X^*} = x^* + \underbrace{U \cap (-x^* + rB_{X^*})}_{w^*\text{-open in } -x^* + rB_{X^*}}$$

as $y^* \mapsto x^* + y^*$ is w^* -homeomorphism, $(x^* + U) \cap rB_{X^*}$ is w^* -open in rB_{X^*} \downarrow

(2) Let $(x_n) \in X$ be a sequence, $x_n \xrightarrow{\|\cdot\|} 0$

$$A = \{x_n, n \in \mathbb{N}\}$$

$$q_A(x^*) = \sup \{|x^*(x_n)|, n \in \mathbb{N}\}$$

Then $\forall c > 0 \{x^* \in X^*; q_A(x^*) < c\}$ is bw^* -open

$\Gamma r > 0 \dots$ Fix $n_0 \in \mathbb{N} \forall n \geq n_0 : \|x_n\| < \frac{c}{2r}$

Hence for $x^* \in rB_{X^*}$ and $n \geq n_0 : |x^*(x_n)| \leq \|x^*\| \|x_n\| < \frac{c}{2}$

Therefore

$$\{x^* \in rB_{X^*}; q_A(x^*) < c\} = \{x^* \in rB_{X^*}; |x^*(x_n)| < c \text{ for } n < n_0\},$$

which is w^* -open in rB_{X^*} \downarrow

③ Let $U \subset X^*$ be w^* -open \checkmark new, $A \subset X$ s.c. $A^0 \cap nB_{X^*} \subset U$
 $\Rightarrow \exists F \subset \frac{1}{n}B_{X^*}$ finite s.t. $(A \cup F)^0 \cap (n+1)B_{X^*} \subset U$

$\Gamma F \subset \frac{1}{n}B_{X^*}$ finite \Rightarrow denote $H_F = (A \cup F)^0 \cap (n+1)B_{X^*} \setminus U$

Then H_F is w^* -compact for any F

($(n+1)B_{X^*} \cap U$ is w^* -open in $(n+1)B_{X^*}$, hence
 $(n+1)B_{X^*} \setminus U$ is w^* -closed in $(n+1)B_{X^*}$,
 thus H_F is w^* -closed)

We claim that $H_F = \emptyset$ for some F . Suppose not.

$$\text{Since } H_{F_1} \cap H_{F_2} = \left((A \cup F_1)^0 \cap (n+1)B_{X^*} \setminus U \right) \cap \left((A \cup F_2)^0 \cap (n+1)B_{X^*} \setminus U \right)$$

$$= \left((A \cup F_1)^0 \cap (A \cup F_2)^0 \right) \cap (n+1)B_{X^*} \setminus U =$$

$$= (A \cup F_1 \cup F_2)^0 \cap (n+1)B_{X^*} \setminus U = H_{F_1 \cup F_2}$$

we see that (H_F, F) has a finite intersection property.

Thus $\bigcap_F H_F \neq \emptyset$. Fix $x^* \in \bigcap_F H_F$.

Then $x^* \in (n+1)B_{X^*} \setminus U$.

Further $\forall F \subset \frac{1}{n}B_{X^*}$ finite $x^* \in F^0$, so

in part. $\exists x \in \frac{1}{n}B_X : |x^*(x)| \leq 1$, so $\|x^*\| \leq n$

so, $x^* \in nB_{X^*}$. Simultaneously $x^* \in A^0$, so

$x^* \in nB_{X^*} \cap A^0 \subset U$, a contradiction \downarrow

(4) $U \subset X^*$ b_w^* open, $0 \in U \Rightarrow \exists A \subset X, A = \{x_n; n \in \mathbb{N}\}$
 with $x_n \rightarrow 0$ s.t. $A^0 \subset U$

$\Gamma: U \cap B_{X^*}$ is rel. b_w^* open in $B_{X^*} \Rightarrow \exists F_1 \subset X$ finite
 s.t. $B_{X^*} \cap F_1^0 = \{x^* \in B_{X^*}; |x^*(x)| \leq 1 \text{ for } x \in F_1\} \subset U$

Using (3) and induction we find for $n \in \mathbb{N}$ a finite set
 $F_{n+1} \subset \frac{1}{n} B_X$ s.t.

$$(F_1 \cup \dots \cup F_{n+1})^0 \cap (n+1) B_{X^*} \subset U$$

Set $A = \bigcup_{n \in \mathbb{N}} F_n$. Then $A = \{x_n\}$, where $x_n \xrightarrow{\|\cdot\|} 0$

$$\text{Moreover, } A^0 = \bigcup_{n \in \mathbb{N}} (A^0 \cap (n+1) B_{X^*}) \subset$$

$$\subset \bigcup_{n \in \mathbb{N}} ((F_1 \cup \dots \cup F_{n+1})^0 \cap (n+1) B_{X^*}) \subset U \quad \square$$

(5) Conclusion: Let \mathcal{T} be the topology of uniform convergence on sequences converging to zero.

By (2) and (1) we see $\mathcal{T} \subset b_w^*$

By (4): Any b_w^* -neighbourhood of 0 is a \mathcal{T} -nbhd of 0
 using (1) we see $b_w^* \subset \mathcal{T}$.