

Theorem VII.12 Let X be a normed space. Then $(X^*, \text{bw}^*)^* = \overline{\mathcal{R}(X)}$

Proof (1) Suppose X is complete, i.e. X is a Banach space

given $(x_n) \subset X$, $x_n \rightarrow 0$, the set $A = \{x_n, n \in \mathbb{N}\} \cup \{0\}$ is norm-compact. Since X is complete, $\overline{\text{aco}} A$ is also norm compact. (see Corollary V.29)

So, it is also weakly compact

Since $\sup_{x \in \overline{\text{aco}} A} |f^*(x)| = \sup_{n \in \mathbb{N}} |f^*(x_n)|$ for each

$f^* \in X^*$, we deduce that $(X^* \subset \text{bw}^* \subset \mu(f^*, X))$

Thus $(X^*, \text{sw}^*)^* = \mathcal{R}(X)$

↑
This $\textcircled{6}$

(2) X is not complete. Let $\hat{X} = \overline{\mathcal{R}(X)}$ denote its completion.

Then $X^* = (\hat{X})^*$ (as X is dense in \hat{X})

Moreover, on $\mathcal{R}B_{X^*}$ the topologies $\sigma(X^*, X)$ and $\sigma(X^*, \hat{X})$ coincide

$\Gamma(\mathcal{R}B_{X^*}, \sigma(X^*, \hat{X}))$ is compact and $\sigma(X^*, X)$ is a weaker Hausdorff topology

It follows that $\text{bw}^*(X^*, X) = \text{bw}^*(X^*, \hat{X})$.

Hence, by (1) we deduce $(X^*, \text{sw}^*) = \overline{\mathcal{R}(X)}$.

Corollary VII.13 X Banach space, ACX^* convex

$\Rightarrow (A \text{ is } w^* \text{-closed} \Leftrightarrow \forall r > 0 \ A \cap rB_{X^*} \text{ is } w^* \text{-closed})$

Pf: By Theorem 12 bw^* is an admissible topology on X^* ,
so by the major theorem a convex set is w^* -closed
iff it is bw^* -closed

Corollary VII.14 X Banach space, $f \in (X^*)^\#$. Then

$f \in \mathcal{A}(X) \Leftrightarrow f \upharpoonright B_{X^*}$ is w^* cts

Pf: \Rightarrow clear

$\Leftarrow f \upharpoonright B_{X^*}$ w^* -cts $\Rightarrow \forall r > 0 \ f \upharpoonright rB_{X^*}$ is w^* -cts
 $\Rightarrow f$ is bw^* -cts $\Rightarrow f \in \mathcal{A}(X)$ (Thm 13)