Proof of Proposition VII.19

X - HLCS, $\dim X = n$

$K \subset X$ compact convex $\Rightarrow$ $\forall x \in K$: x is a convex combination
of points $x_1, \ldots, x_k \in X$ with properties:

- $x_1, \ldots, x_k \in \text{ext } K$
- $x_1, \ldots, x_k$ are affinely independent
- $k \leq n+1$

Recall: $x_1, \ldots, x_k$ are affinely independent $\Leftrightarrow$ $x_2 - x_1, \ldots, x_k - x_1$ are
linearly independent.

Proof ① $n = 0 \Rightarrow k = \{0\}$ $\Big\}$ clear
$n = 1 \Rightarrow k = [a, b] \subset \mathbb{R}$ or $k = \{a\} \subset \mathbb{R}$

② Suppose $n \geq 1$ and that it is true if $\dim X \leq n$.
Suppose $\dim X = n+1$

- WLOG $0 \in K$ (replace K by $K - x$ for some $x \in K$)
- WLOG $\text{span } K = X$ (otherwise replace X by $\text{span } K$
  and apply the induction hypothesis
  for $\dim (\text{span } K) \leq n$)

- Then K has nonempty interior
  Let $x_1, \ldots, x_{n+1} \in K$ be linearly independent,
  hence a basis of X
  Then $T : \mathbb{R}^{n+1} \to X$ defined by
  $T(t_1, \ldots, t_{n+1}) = \sum_{j=1}^{n+1} t_j \cdot x_j$ is an isomorphism
  of $\mathbb{R}^{n+1}$ onto X
  Then we deduce that $\text{co} \{0, x_1, \ldots, x_{n+1}\}$ has
  nonempty interior, as in $\mathbb{R}^{n+1}$
  $\text{co} \{0, e^1, \ldots, e^{n+1}\}$ has nonempty interior

  So, $\text{Int } K \neq \emptyset$ $\Big\downarrow$

- Let $x \in K$. If $x \in \partial K$, then $x \notin \text{Int } K$,

so $\exists\, f \in X^* \setminus \{0\}$ $f(x) \geq \sup f(\text{Int } K) = \sup f(K)$

(by Theorem $\mathcal{I}$.36 (a))

So, $F = \{ y \in K,\ f(y) = f(x) \}$

is a closed face

We can use inductive hypothesis

on F (its dimension is at most n) [and $\angle 17$ (a, s)]

	Int K is dense in K
	(see Lemma $\mathcal{I}$.16)

- $x \in \text{Int } K$. Fix $y \in \text{ext} K$ arbitrary. Consider the

line $y + t(x-y),\ t \in \mathbb{R}$. This line intersects K in a segment

one endpoint is y. Denote the second one by z. Then

$z \in \partial K$. Use the previous case to deduce that

$z = t_1 a_1 + \cdots + t_k a_k,\quad a_1, \ldots, a_k \in \text{ext} K$ (in fact,

they are in $\text{ext} F$, where F is the face constructed

in the previous case), $k \leq n$, $a_1, \ldots, a_k$

affinely independent.

Since $x = (1-s) y + s z$ for some $s \in (0,1)$, we have

$x = (1-s) y + s t_1 a_1 + \cdots + s t_k a_k$

It remains to observe that $y, a_1, \ldots, a_q$ are affinely independent.

⌈ Let $t_0 (y - a_1) + t_2 (a_2 - a_1) + \cdots + t_k (a_q - a_1) = 0$

Apply the functional f constructed in the first case

then $f(a_q) = f(a_2) = \cdots = f(a_k) > f(y)$

Then $t_0 (f(y) - f(a_1)) = 0$, so $t_0 = 0$

Since $a_1, \ldots, a_q$ are aff. independent, we see

$$a_2 = \cdots = a_q = 0 \qquad \rfloor$$