PROOF OF PROPOSITION VII.21 (MILMAN THEOREM)

X HLCS, $K \subset X$ compact convex, $A \subset K$, $K = \overline{\text{co}\, A}$

$\Rightarrow$ ext $K \subset \overline{A}$

① If $A \subset K$ is such that $K = \text{co}\, A$, then ext $K \subset A$

[by the very definition of extreme points]

② Let U be an absolutely convex open nbhd of 0 in X.

Then there is a finite set $F \subset \overline{A}$ with $F + U \supset \overline{A}$

(using compactness of $\overline{A}$).

Then $K = \overline{\text{co}\, A} \subset \overline{\text{co}\,((F+U) \cap K)} =$

$= \overline{\text{co}\left(\bigcup_{x \in F} (x+\overline{U}) \cap K\right)} = \text{co}\left(\bigcup_{x \in F} (x+\overline{U}) \cap K\right)$

$(x+\overline{U}) \cap K$, $x \in F$ are compact convex sets, and they are finitely many, so the convex hull of their union is compact

(see the proof of Lemma XI.2)

Thus by ① we get ext $K \subset \bigcup_{x \in F} (x+\overline{U}) \cap K \subset \overline{A} + \overline{U}$

Since U is arbitrary, we get ext $K \subset \overline{A}$ and we are done

[$x \notin \overline{A} \Rightarrow \exists\, U$ absolutely convex open nbhd of 0 s.t.

$(x+U) \cap \overline{A} = \emptyset$ then $(x + \frac{1}{2}\overline{U}) \cap \overline{A} = \emptyset$, so

$x \notin \overline{A} + \frac{1}{2}\overline{U}$]