SEVERAL EXAMPLES ON EXTREME POINTS

(1) $X = \ell_1$, $K = B_X$, the closed unit ball

- $\operatorname{ext} K = \{\pm e^n, n \in \mathbb{N}\}$ (in the real case):
  - $e^n \in \operatorname{ext} K$: $e^n = \frac{1}{2}(x+y)$, $x, y \in B_X$

    look at the n-th coordinate: $1 = \frac{1}{2}(x_n + y_n)$

    Since $x_n \leq 1$ and $y_n \leq 1$, necessarily $x_n = y_n = 1$.

    But since $x, y \in B_X$, necessarily $x = y = e^n$

  - $-e^n \in \operatorname{ext} K$ similarly
  - $x \in K \setminus \{\pm e^n, n \in \mathbb{N}\}$

    either $\|x\| < 1$. Then x is an interior point of a segment in K: $x = 0$ ... take $[-e^1, e^1]$

    $x \neq 0$ ... take $\left[0, \frac{x}{\|x\|}\right]$

    or $\|x\| = 1$. Then there are m, n, $m < n$ s.t. $x_m \neq 0$, $x_n \neq 0$

    Fix $\varepsilon > 0$, $\varepsilon < \min(|x_m|, |x_n|)$

    If $\operatorname{sgn} x_m = \operatorname{sgn} x_n$ define y, z as follows

$$z_k = \begin{cases} x_m + \varepsilon, & k = m \\ x_n - \varepsilon, & k = n \\ x_k & \text{otherwise} \end{cases} \qquad y_k = \begin{cases} x_m - \varepsilon & k = m \\ x_n + \varepsilon & k = n \\ x_k & \text{otherwise} \end{cases}$$

Then $y, z \in B_X$, $\|y\| = \|z\| = 1$, $y \neq z$, $x = \frac{y+z}{2}$

so $x \notin \text{ext}\, K$

If $\text{sgn}\, x_m = -\text{sgn}\, x_n$ define y, z as follows:

$$z_k = \begin{cases} x_m + \varepsilon, & k = m \\ x_n + \varepsilon, & k = n \\ x_k & \text{otherwise} \end{cases} \qquad y_k = \begin{cases} x_m - \varepsilon & k = m \\ x_n - \varepsilon & k = n \\ x_k & \text{otherwise} \end{cases}$$

Then $y, z \in B_X$, $\|y\| = \|z\| = 1$, $y \neq z$, $x = \frac{y+z}{2}$

so $x \notin \text{ext}\, K$

- X is a dual space, $\ell_1 = c_0^*$, then B_X is w^*-compact, Hence, Krein–Milman shows $K = \overline{\text{co}}^{w^*}\, \text{ext}\, K$

- Directly: $\text{co}\, \text{ext}\, K = \{x \in K;\ \{n;\ x_n \neq 0\}$ is finite$\}$

  $\subset$: clear from the description of $\text{ext}\, K$ above

  $\supset$: $\{n;\ x_n \neq 0\} = \{n_1 < n_2 < \dots < n_k\}$

  Assume $k \geq 1$ (i.e. $x \neq 0$). Then
  $$|x_{n_1}| + \dots + |x_{n_k}| \leq 1$$

  If the equality holds, then
  $$x = |x_{n_1}| \cdot (\text{sgn}\, x_{n_1}) e^{n_1} + \dots + |x_{n_k}| (\text{sgn}\, x_{n_k}) e^{n_k},$$

So $x \in \text{co ext } K$

If $|x_{n_1}| + \cdots + |x_{n_k}| < 1$, let $t = 1 - (|x_{n_1}| + \cdots + |x_{n_k}|)$

then $x = \frac{t}{2} \cdot (-\text{sgn}\, x_{n_1} \cdot e^{n_1}) + (|x_{n_1}| + \frac{t}{2}) \cdot \text{sgn}\, x_{n_1} e^{n_1}$

$+ |x_{n_2}| \text{ sgn}\, x_{n_2} \cdot e^{n_2} + \cdots + |x_{n_k}| \text{ sgn}\, x_{n_k} e^{n_k}$

So, $x \in \text{co ext } K$

Finally, if $x = 0$, then $x = \frac{1}{2}(e^1 + (-e^1))$
$\in \text{co ext } K$

- Hence, we have even $K = \overline{\text{co ext} K}^{\|\cdot\|}$

$\Big(x \in K \Rightarrow x = \lim\limits_{n \to \infty} (x_1, \dots, x_n, 0, 0, \dots)$ (in $\|\cdot\|$)

and $(x_1, \dots, x_n, 0, 0, \dots) \in \text{co ext } K \Big)$

(2) $X = C(K)^*$, where K is a compact space

$A = P(K)$, the probability measures

- $\text{ext } A = \{\delta_x,\ x \in K\}$

- $x \in K \Rightarrow \delta_x \in \text{ext } P(K)$:

Assume $\delta_x = \frac{1}{2}(\mu_1 + \mu_2)$, $\mu_1, \mu_2 \in P(K)$

Evaluate at x:

$$1 = \frac{1}{2}\left(\mu_1(\{x\}) + \mu_2(\{x\})\right)$$

Since $\mu_1(\{x\}), \mu_2(\{x\}) \in [0,1]$,

we deduce that $\mu_1(\{x\}) = \mu_2(\{x\}) = 1$

Since μ_1, μ_2 are probabilities, $\mu_1 = \mu_2 = \delta_x$

Thus, indeed, $\delta_x \in \operatorname{ext} P(K)$

- $\mu \in P(K) \setminus \{\delta_x ; x \in K\}$

Then there are $B_1, B_2 \subset K$ disjoint Borel sets s.t. $\mu(B_1) > 0$, $\mu(B_2) > 0$

Let $\mathcal{U} = \{U \subset K \text{ open} ; \mu(U) = 0\}$

Let $G := \bigcup \mathcal{U}$. Then G is open. Moreover, $\mu(G) = 0$ [$L \subset G$ compact $\Rightarrow \exists\, \mathcal{F} \subset \mathcal{U}$ finite s.t. $L \subset \bigcup \mathcal{F}$. Thus $\mu(L) = 0$

μ Radon $\Rightarrow \mu(G) = 0$]]

$H := K \setminus G \Rightarrow \mu(H) = 1$. μ is not Dirac $\Rightarrow$ $\exists\, x, y \in H$ $x \neq y$. Let U, W be disjoint open neighborhoods of x, y.

Then $\mu(V) > 0$, $\mu(W) > 0$ $\left(\begin{array}{l} \text{as } V \not\subset G \\ \quad W \not\subset G \end{array}\right)$

Let $\delta, \eta \in (0,1)$.

Define ν_1, ν_2 by:

$$\nu_1(B) = \mu(B \setminus (B_1 \cup B_2)) + (1+\delta)\mu(B \cap B_1) + (1-\eta)\mu(B \cap B_2)$$

$$\nu_2(B) = \mu(B \setminus (B_1 \cup B_2)) + (1-\delta)\mu(B \cap B_1) + (1+\eta)\mu(B \cap B_2)$$

Then ν_1, ν_2 are positive measures, $\frac{1}{2}(\nu_1 + \nu_2) = \mu$,

$\nu_1 \neq \nu_2$. By a suitable choice of δ, η

we get $\nu_1, \nu_2 \in P(K)$:

It is enough: $\delta\mu(B_1) = \eta\mu(B_2)$

and this may be achieved.

Thus $\mu \notin \text{ext}\, P(K)$

- X is a dual space, $P(K)$ is w^*-compact

$(P(K) = \{\mu \in B_{C(K)^*};\ \mu(1) = 1\})$

So, by Krein-Milman $P(K) = \overline{\text{co ext}\, P(K)}^{w^*}$

- There is also a norm-topology:

co ext $P(K)$ = finitely supported probabilities

[clear from the above representation]

$\overline{\text{co ext } P(K)}^{\|\cdot\|}$ = countably supported probabilities

$\supset$: $\mu = \sum_{n=1}^{\infty} t_n \delta_{x_n} \Rightarrow \mu = \lim_{N\to\infty} \left(\sum_{n=1}^{N} t_n \delta_{x_n} + \left(\sum_{n=N+1}^{\infty} t_n \right) \delta_{x_{N+1}} \right)$

$t_n \geq 0, \sum t_n = 1$

the limit in norm;

$\subset$: Countably supported probabilities form a norm-closed set, because it is $P(K) \cap$ ctbly supported measures and ctbly supported measures form a subspace isometric to $\ell_1(K)$, hence norm-closed.

(3) $X = \ell_\infty$, $K = B_X$ (the real version)

- let $K = \{ (x_n) ; \forall n : x_n = 1 \text{ or } x_n = -1 \}$

$\supset$: $x = \frac{1}{2}(y+z)$, $y, z \in K$

$\Rightarrow \forall n: x_n = \frac{1}{2}(y_n + z_n)$

Since $y_n, z_n \in [-1,1]$ and $x_n = 1$ or $x_n = -1$,

clearly $y_n = z_n = x_n$.

So $y = z = x$, hence $x \in \text{ext} K$

$\subset$: Assume $\exists n: x_n \neq 1$ and $x_n \neq -1$, so $x_n \in (-1,1)$

Find $\varepsilon > 0$ s.t. $[x_n - \varepsilon, x_n + \varepsilon] \subset [-1,1]$

Let y, z be defined by

$$y_k = \begin{cases} x_n + \varepsilon, & k = n \\ x_k & \text{otherwise} \end{cases} \qquad z_k = \begin{cases} x_n - \varepsilon, & k = n \\ x_k & \text{otherwise} \end{cases}$$

$\Rightarrow y, z \in K$, $y \neq z$, $\frac{1}{2}(y+z) = x$, so $x \notin \text{ext} K$

- X is a dual space, $\ell_\infty = (\ell_1)^*$, K is w^*-compact.

Thus Krein-Milman $\Rightarrow$ $K = \overline{\text{co}\, \text{ext} K}^{w^*}$

- Norm topology: First, compute $\text{co}\, \text{ext} K$

$\mathrm{co}\,\partial K = \{ x \in K ; \ \{ t_n, n \in \mathbb{N} \} \text{ is finite} \}$

(i.e. sequences which have only finitely many values)

$\subset$: $\quad x = t_1 y^1 + \dots + t_k y^k, \ y^1, \dots, y^k \in \partial K$

Then $\forall n$: $x_n \in \{ \pm t_1 \pm t_2 \pm \dots \pm t_k \}$,

where all combinations of signs are allowed

This shows that we have at most 2^k values.

$\supset$: Assume that $x \in K$ attains finitely many values

$x = (x_n)$

and $\{ x_n, n \in \mathbb{N} \} = \{ a_1 < a_2 < \dots < a_k \}$

Let $A_j = \{ n \in \mathbb{N} ; \ x_n = a_j \}$

We know that $a_j \in [-1, 1]$,

so $a_j = (1 - t_j) \cdot (-1) + t_j \cdot 1$,

where $t_j = \dfrac{1 + a_j}{2}$

Then $0 \le t_1 < t_2 < \dots < t_k \le 1$

Set $s_1 = t_1, \ s_2 = t_2 - t_1, \dots, s_k = t_k - t_{k-1}, \ s_k = 1 - t_k$

then $s_1,\dots,s_{k+1}\geq 0$, $\quad s_1+\dots+s_{k+1}=1$

Define $y^1,\dots,y^{k+1}\in \mathrm{ext}\,K$ as follows:

$y^1_n = 1$, $n\in\mathbb{N}$

$y^2_n = \begin{cases} -1, & n\in A_1 \\ 1 & \text{otherwise}\end{cases}$

$y^3_n = \begin{cases} -1, & n\in A_1\cup A_2 \\ 1, & \text{otherwise}\end{cases}$

$\vdots$

$y^{k+1}_n = -1$, $n\in A_1\cup\dots\cup A_k = \mathbb{N}$

Then $x = s_1y^1+s_2y^2+\dots+s_{k+1}y^{k+1}$

$\Big[$ $n\in\mathbb{N}$ ... $\exists!\, j : n\in A_j$

$$s_1y^1_n+\dots+s_jy^j_n+s_{j+1}y^{j+1}_n+\dots+s_{k+1}y^{k+1}_n$$

$$= s_1+\dots+s_j - s_{j+1}-\dots-s_{k+1}$$

$$= t_j - (1-t_j) = a_j = x_n \Big]$$

- $\overline{\mathrm{co}\,\mathrm{ext}\,K}^{\|\cdot\|} = K$: Any $x\in K$ can be approximated by elements attaining finitely many values

$x \in K$, $n \in \mathbb{N}$

define $A_j = \{ n \in \mathbb{N} ;\ x_n \in [-1+\frac{j}{n}, -1+\frac{j+1}{n}) \}$

$j = 0, 1, \dots, 2n$

define $y_n = -1 + \frac{j}{n}$, $n \in A_j$

Then $y \in \mathrm{co}\,\mathrm{ext}\,K$ and $\|y - x\| \leq \frac{1}{n}$

$n \in \mathbb{N}$ arbitrary $\Rightarrow x \in \overline{\mathrm{co}\,\mathrm{ext}\,K}^{\|\cdot\|}$

- Alternative proof that $\mathrm{co}\,\mathrm{ext}\,K$ = elements attaining only finitely many values, namely of a^{j}".

Define a_j and A_j as above.

Note that $a = (a_1, \dots, a_k) \in B_Z$, where

$Z = (\mathbb{R}^k, \|\cdot\|_\infty)$ $(= \ell_\infty^k)$.

By Minkowski - Caratéodory thm

$B_Z = \mathrm{co}(\mathrm{ext}\,B_Z)$, more precisely

$a = t_1 b^1 + \dots + t_{k+1} b^{k+1}$, where $b^1, \dots, b^{k+1} \in \mathrm{ext}\,B_Z$

define $y_n^j = b_m^j$ if $n \in A_m$, $j = 1, \dots, k+1$

Then $y^j \in \text{ext}\, K$, $x = t_1 y^1 + \dots + t_{k+1} y^{k+1}$