Proof of Lemma VII.25

Let T be an angelic space

(1) $A \subset T$ compact $\Rightarrow$ A sequentially compact

⌈ Let $(x_n) \subset A$ be a sequence. If it has a constant subsequence, it has a convergent subsequence.
If it has no constant subsequence, it has a one-to-one subsequence. So, assume (x_n) is one-to-one.
Set $B = \{x_n, n \in \mathbb{N}\}$. Then B is relatively countably compact; so, it has a cluster point $x \in T$. Then $x \in \overline{B \setminus \{x\}}$. Since $B \setminus \{x\}$ is also relatively ctbly compact, by property (ii) there is a sequence (n_k) with $x_{n_k} \neq x$, $x_{n_k} \to x$

Up to passing to a subsequence (n_k) is increasing

(2) So, for $A \subset T$ we have

A rel. ctbly compact $\Leftrightarrow$ A rel. compact $\Leftrightarrow$ rel. sequentially compact

$\Rightarrow$ by property (i)
$\Leftarrow$ trivial

$\Rightarrow$ by (1)
$\Leftarrow$ by property (i)

(3) $A \subset T$ ctbly compact $\Rightarrow$ A compact

⌈ By property (i) we deduce that $\overline{A}$ is compact.
We will show that $A = \overline{A}$. Fix $x \in \overline{A}$. By property (ii) there is a sequence $(x_n) \subset A$ with $x_n \to x$. Since A is ctbly compact, the sequence (x_n) has a cluster point in A. But its only cluster point is x, so $x \in A$ ⌋

(4) Using (1) and (3) it follows that for $A \subset T$

A is ctbly compact $\Leftrightarrow$ A is compact $\Leftrightarrow$ A is sequentially compact