

Proof of Lemma VII.25

Let T be an angelic space

(1) ACT compact $\Rightarrow A$ sequentially compact

Γ Let $(x_n) \subset A$ be a sequence. If it has a constant subsequence, it has a convergent subsequence.

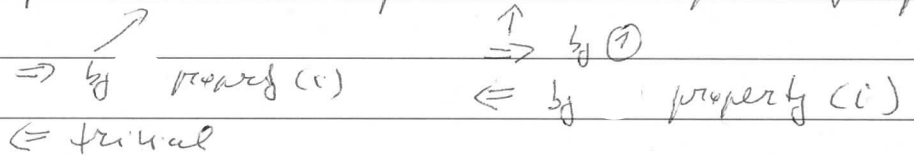
If it has no constant subsequence, it has a one-to-one subsequence. So, assume (x_n) is one-to-one.

Set $B = \{x_n, n \in \mathbb{N}\}$. Then B is relatively countably compact. So, it has a cluster point $x \in T$. Then

$x \in \overline{B} \setminus \{x\}$. Since $B \setminus \{x\}$ is also relatively countably compact, by property (cc) there is a sequence (x_{n_k}) with $x_{n_k} \neq x, x_{n_k} \rightarrow x$

(2) So, for ACT we have Up to passing to a subsequence (x_{n_k}) is increasing

A rel. ctly compact $\Leftrightarrow A$ rel. compact $\Leftrightarrow A$ rel. sequentially compact



(3) ACT ctly compact $\Rightarrow A$ compact

Γ by property (c) we deduce that \overline{A} is compact.

We will show that $A = \overline{A}$. Fix $x \in \overline{A}$. By property (cc) there is a sequence $(x_n) \subset A$ with $x_n \rightarrow x$. Since A is ctly compact, the sequence (x_n) has a cluster point in A . But it's only cluster point is x , so $x \in A$.

(4) Using (1) and (3) it follows that for ACT

A is ctly compact $\Leftrightarrow A$ is compact $\Leftrightarrow A$ is sequentially compact