Proof of Lemma VII.27: Let K be compact, $A \subset C(K)$ τ_p-relatively ctbly compact. Then A is rel. τ_p-compact

(1) $x \in K \Rightarrow \{f(x);\ f \in A\}$ is bdd in $\mathbb{F}$

[If not, there is $(f_n) \subset A$ with $|f_n(x)| \to \infty$. But then (f_n) has no τ_p-cluster point]

(2) $r(x) := \sup\{|f(x)|;\ f \in A\}$, $x \in K$

By (1) $r(x) < \infty$ for $x \in K$, so

$$A \subset \{f \in C(K);\ |f(x)| \le r(x) \text{ for } x \in K\} \subset$$
$$\subset \{f \in \mathbb{F}^K;\ |f(x)| \le r(x) \text{ for } x \in K\} =$$
$$= \prod_{x \in K} \{\lambda \in \mathbb{F};\ |\lambda| \le r(x)\}$$

The last set is compact in $\mathbb{F}^K$ by Tychonoff theorem.

So, it is enough to prove that $\overline{A}^{\mathbb{F}^K} \subset C(K)$

(3) Fix $f \in \overline{A}^{\mathbb{F}^K}$. Suppose that $f \notin C(K)$.

Fix $x \in K$ s.t. f is not continuous at x

Fix $\varepsilon > 0$ s.t. $\forall U \ni x$, a neighborhood $\exists y \in U: |f(y) - f(x)| \ge \varepsilon$

Let us construct by induction $y_n \in K$, U_n neighborhoods of x and $f_n \in A$ for $n \in \mathbb{N}$ such that

[i] $|f_n(x) - f(x)| < \frac{1}{n}$, $n \in \mathbb{N}$

[ii] $|f_n(y_k) - f(y_k)| < \frac{1}{n}$ for $n \in \mathbb{N}$, $k < n$

[iii] $U_n := \{z \in K;\ |f_k(z) - f_k(x)| < \frac{1}{n} \text{ for } k \le n\}$

[iv] $y_n \in U_n$ & $|f(y_n) - f(x)| \ge \varepsilon$

[Construction: Find f_1 s.t. [i] holds (as $f \in \overline{A}^{\mathbb{F}^K}$)

Given $f_1, \dots, f_n$, $y_1, \dots, y_{n-1}$ ... define U_n by [iii], find y_n as in [iv] (U_n is a neighborhood of x); find f_n satisfying [i], [ii] (as $f \in \overline{A}^{\mathbb{F}^K}$)

K compact $\Rightarrow \exists\, y \in K$, a cluster point of (y_n)

A rel. ctbly compact $\Rightarrow \exists\, g \in \mathcal{C}(K)$, a τ_p-cluster point of (f_n)

By [ii] : $\forall k \in \mathbb{N}$: $f_n(y_k) \to f(y_k)$
Since $g(y_k)$ is a cluster point of $(f_n(y_k))_n$,
we deduce $f(y_k) = g(y_k)$, $k \in \mathbb{N}$

By [iv] and [iii] : $y_n \in U_n$, so $\forall k \in \mathbb{N}$: $f_k(y_n) \to f_k(x)$
Since f_k is cts and y is a cluster point of (y_n), $f_k(y)$ is a cluster point of $(f_k(y_n))_n$
Therefore, $f_k(y) = f_k(x)$, $k \in \mathbb{N}$

By [i] : $f_k(x) \to f(x)$, so $f_k(y) \to f(x)$.
Since g is a τ_p-cluster point of f_k,
$g(y)$ is a cluster point of $(f_k(y))$, thus $g(y) = f(x)$

Finally, as $f(y_k) = g(y_k)$, g is cts and y is a cluster point of (y_k), we deduce that $g(y)$ is a cluster point of $(g(y_k))$

Therefore, $\underbrace{f(x)}_{= g(y)}$ is a cluster point of $\underbrace{f(y_k)}_{= g(y_k)}$

But by [iv] $|f(x) - f(y_k)| > \varepsilon$ for $k \in \mathbb{N}$, a contradiction.

Proof of Theorem VII.28 K compact Hausdorff space

$A \subset C(K)$, $f \in C(K)$

$$f \in \overline{A}^{\tau_p} \Rightarrow \exists\, C \subset A \text{ ctble } f \in \overline{C}^{\tau_p}$$

① $f \in \overline{A}^{\tau_p} \iff \forall x_1, \dots, x_k \in K\ \forall n \in \mathbb{N}\ \exists g \in A:$
$$|g(x_j) - f(x_j)| < \tfrac{1}{n} \quad j = 1 \dots k$$

② For $k, n \in \mathbb{N}$ and $g \in C(K)$ define

$$U_{k,n}(g) = \{(x_1, \dots, x_k) \in K^k;\ |g(x_j) - f(x_j)| < \tfrac{1}{n} \quad j = 1 \dots k\}$$

Then $U_{k,n}(g)$ is an open subset of K^k

and
$$f \in \overline{A}^{\tau_p} \iff \forall k, n \in \mathbb{N}: \bigcup_{g \in A} U_{k,n}(g) = K^k \qquad (*)$$

[This is just a reformulation of the equivalence in ①]

③ The proof:

$$f \in \overline{A}^{\tau_p} \overset{(*)}{\Rightarrow} \bigcup_{g \in A} U_{k,n}(g) = K^k \text{ for any } k, n \in \mathbb{N}$$

Fix k, n. Since K^k is compact, there is a finite set

$$F_{k,n} \subset A \text{ s.t. } \bigcup_{g \in F_{k,n}} U_{k,n}(g) = K^k$$

Set $C := \bigcup_{k,n \in \mathbb{N}} F_{k,n}$. Then $C \subset A$ is ctble and

$$\forall k, n \in \mathbb{N}: \bigcup_{g \in C} U_{k,n}(g) = K^k \overset{(*)}{\Rightarrow} f \in \overline{C}^{\tau_p}$$

Proposition VII.29 K compact Hausdorff, $A \subset C(K)$
(A, τ_p) compact & separable $\Rightarrow$ (A, τ_p) metrizable

Proof: ① If K is metrizable, then it is separable. Fix $D \subset K$ ctble dense set. Then the topology $\tau_p(D)$ [see Example 11.2 (5)] is metrizable (see Theorem 1.22).
On A $\tau_p(D)$ coincides with τ_p, since A is compact and $\tau_p(D)$ is a weaker Hausdorff topology.
Thus (A, τ_p) is metrizable

② K general: Since A is τ_p-separable, fix a ctble dense set $\{f_n, n \in \mathbb{N}\}$

Define $\varphi : K \to \mathbb{F}^{\mathbb{N}}$ by $\varphi(t) = (f_n(t))_{n=1}^{\infty}$, $t \in K$

Then φ is cts, hence $\varphi(K) =: L$ is compact. Since $\mathbb{F}^{\mathbb{N}}$ is metrizable, L is metrizable

Define $\varphi^* : C(L) \to C(K)$ by setting $\varphi^*(f) = f \circ \varphi$, $f \in C(L)$

Then φ^* is a linear isometry of $C(L)$ into $C(K)$
[linearity is clear.
$$\|\varphi^*(f)\|_\infty = \|f \circ \varphi\|_\infty = \sup_{t \in K} |f(\varphi(t))| = \sup_{y \in \varphi(K)} |f(y)| = \|f\|_\infty$$
$\uparrow$ $\varphi(K) = L$]

Moreover, φ^* is $\tau_p - \tau_p$ homeomorphism:

continuity: $x \in K \Rightarrow \varphi^*(f)(t) = f(\varphi(t))$ and $f \mapsto f(\varphi(t))$ is τ_p-cts

continuity of the inverse: $y \in L$ .... fix some $t \in K$ with $\varphi(t) = y$
Then $\forall g \in \varphi^*(C(L))$:
$$(\varphi^*)^{-1}(g)(y) = g(t),$$ so it is τ_p-cts

Further, $\varphi^*(\mathcal{C}(L))$ is τ_p-closed in $\mathcal{C}(K)$, as

$$\varphi^*(\mathcal{C}(L)) = \{f \in \mathcal{C}(K);\ \forall x, y \in K : \varphi(x) = \varphi(y) \Rightarrow f(x) = f(y)\}$$

- clearly: the set on RHS is τ_p-closed
- clearly: "$\subset$" holds
- $\supset$: $f \in$ RHS. Define $g : L \to \mathbb{F}$ by $g(\varphi(t)) = f(t)$, $t \in K$. It is clearly well defined. It remains to show that g is continuous:

  $H \subset \mathbb{F}$ closed $\Rightarrow g^{-1}(H) = \varphi(f^{-1}(H))$

  $f^{-1}(H)$ is closed (by continuity of f), hence compact. Thus $\varphi(f^{-1}(H))$ is compact, hence closed.

Finally, $f_n \in \varphi^*(\mathcal{C}(L))$, as $f_n = \pi_n \circ \varphi$, where $\pi_n : L \to \mathbb{F}$ is the projection onto n-th coord.

So, $A \subset \varphi^*(\mathcal{C}(L))$.

It follows that A is homeomorphic to a subset of $(\mathcal{C}(L), \tau_p)$ [namely to $(\varphi^*)^{-1}(A)$]. Thus A is metrizable by ①

As a corollary we get Theorem 26:

(a) K compact Hausdorff $\Rightarrow$ $(C(K), \tau_p)$ is angelic

$\ulcorner$ $A \subset (C(K), \tau_p)$ rel. ctbly compact $\Rightarrow$ $\overline{A}^{\tau_p}$ is compact by Lemma 27,

so (i) is satisfied.

$f \in \overline{A}^{\tau_p} \overset{\text{Thm 28}}{\Longrightarrow} \exists\, C \subset A$ ctble $f \in \overline{C}^{\tau_p}$. The $\overline{C}^{\tau_p}$ is compact separable, thus metrizable (by Prop. 29)

It follows that there is $(f_n) \subset C$ s.t. $f_n \xrightarrow{\tau_p} f$ $\lrcorner$

(b) X Banach space $\Rightarrow$ (X, w) angelic

$\ulcorner$ $(X, w) \subset C((B_{X^*}, w^*), \tau_p)$ by Theorem 15 $\lrcorner$