Proof of Theorem VI.33 Let X, Y be Banach spaces and $T \in \mathcal{L}(X,Y)$

(i) $\Rightarrow$ (ii) T weakly compact $\Rightarrow$ T' weakly compact

Suppose T is weakly compact. Set $L := \overline{T(B_X)}$. Then L is weakly compact.

Define $R: Y^* \to \mathcal{C}(L)$ by $R(y^*) = y^*|_L$, $y^* \in Y^*$

Then ① R is a linear operator

② $\forall y^* \in Y^*$: $\|R(y^*)\|_\infty = \|T'y^*\|_{X^*}$

$$\Big[\ \|T'y^*\|_{X^*} = \sup_{x \in B_X} |T'y^*(x)| = \sup_{x \in B_X} |y^*(Tx)| =$$

$$= \sup_{y \in T(B_X)} |y^*(y)| \underset{\overline{T(B_X)} = L}{=} \sup_{y \in L} |y^*(y)| = \|R(y^*)\|_\infty \ \Big]$$

③ It follows that $\|R\| = \|T'\|$ $(= \|T\|)$, so, in particular R is bdd.

Moreover, there is an isometry $S : R(Y^*) \to T'(Y^*)$ such that $T' = S \circ R$

④ Clearly, R is $w^* \to \tau_p$ continuous. Hence $R(B_{Y^*})$ is τ_p-compact in $\mathcal{C}(L)$. Since it is bdd (see ③), it is even weakly compact (by Thm 31)

Since S is an isometry, it is w-w compact.

Thus $T'(B_{Y^*}) = S(R(B_{Y^*}))$ is weakly compact.

Hence, T' is weakly compact

(ii) $\Rightarrow$ (iii) T' weakly compact $\Rightarrow$ T' is $w^* \to w$ cts

[T' weakly compact $\Rightarrow$ $T'(B_{Y^*})$ is weakly compact, hence on $T'(B_{Y^*})$ weak and weak* topologies coincide [w^*-topology is a weaker Hausdorff topology]

Since T' is w^*-w^* cts (as any dual operator), $T'\restriction_{B_{Y^*}}$ is w^*-w cts.

So, given any $x^{**} \in X^{**}$: $x^{**} \circ T'\restriction_{B_{Y^*}}$ is w^*-cts,

thus $x^{**} \circ T'$ is w^*-cts [Banach–Dieudonné]

So, T' is $w^* \to w$ cts]

(iii) $\Rightarrow$ (iv) T' $w^* \to w$ cts $\Rightarrow$ $T''(X^{**}) \subset \varkappa(Y)$

[Suppose T' is w^*-w cts. Fix $x^{**} \in X^{**}$.
Then $T''(x^{**}) = x^{**} \circ T'$ is w^*-cts.
So, it belongs to $\varkappa(Y)$ by Section 11.1]

(iv) $\Rightarrow$ (i) $T''(X^{**}) \subset \varkappa(Y)$ $\Rightarrow$ T is weakly compact

[T'' is $w^* \to w^*$ (i.e. $\sigma(X^{**}, X^*) \to \sigma(Y^{**}, Y^*)$) cts, as a dual operator.

It follows that $T''(B_{X^{**}})$ is $\sigma(Y^{**}, Y^*)$-compact.

Since $T''(B_{X^{**}}) \subset \varkappa(Y)$, it is $\sigma(\varkappa(Y), Y^*)$-compact, thus weakly compact.

As $\varkappa_Y(T(B_X)) \subset T''(\varkappa_X(B_X)) \subset T''(B_{X^{**}})$, we deduce that $\overline{\varkappa_Y(T(B_X))}$ is weakly compact, thus $\overline{T(B_X)}$ is weakly compact]