SEVERAL EXAMPLES

(1) $[0,1]^\Gamma$ (or $\{0,1\}^\Gamma$) is compact for any set Γ. This follows from Tychonoff theorem.

(2) Let $\Gamma = \{0,1\}^{\mathbb{N}}$ (the set of all sequences of 0s and 1s). Then $\{0,1\}^\Gamma$ is compact (by (1)) but not sequentially compact:

[We define a sequence $(f_n) \subset \{0,1\}^\Gamma$ as follows:

$$f_n(\gamma) = \text{the } n\text{-th element of } \gamma, \quad \gamma \in \Gamma$$

[note that $\gamma \in \Gamma = \{0,1\}^{\mathbb{N}}$, so γ is a sequence and its elements are 0 or 1]]

Then (f_n) is indeed a sequence in $\{0,1\}^\Gamma$, but no subsequence is convergent:

Assume (n_k) is an increasing sequence of natural numbers. Then the sequence (f_{n_k}) has no limit in $\{0,1\}^\Gamma$.
To this end it is enough to find $\gamma \in \Gamma$ such that $(f_{n_k}(\gamma))$ has no limit in $\{0,1\}$, as the convergence is coordinatewise.

So, let us find such a sequence γ.

We set $\gamma_m = \begin{cases} 1 & \text{if } m = n_k, \ k \text{ even} \\ 0 & \text{if } m = n_k, \ k \text{ odd} \\ 0 & \text{if } m \neq n_k \text{ for any } k \end{cases}$

Then $f_{n_k}(\gamma) = \begin{cases} 1 & k \text{ even} \\ 0 & k \text{ odd} \end{cases} \Rightarrow$ it has no limit.]

(1) Let Γ be an uncountable set. Then

$$A = \{ x \in [0,1]^\Gamma ; \{ \gamma \in \Gamma ; x(\gamma) \neq 0 \} \text{ is countable} \}$$

is sequentially compact and it is a proper dense subset of $[0,1]^\Gamma$.

- $A \subsetneq [0,1]^\Gamma$ as Γ is uncountable

- A is dense in $[0,1]^\Gamma$: Let $x \in [0,1]^\Gamma$ be arbitrary. Let U be a neighborhood of x.
  Then there are $\gamma_1, \dots, \gamma_n \in \Gamma$, $\varepsilon > 0$ s.t.

  $$x \in \{ y \in [0,1]^\Gamma ; |y(\gamma_j) - x(\gamma_j)| < \varepsilon \text{ for } j = 1 \dots n \} \subset U$$

  Let $y \in [0,1]^\Gamma$ be defined by:

  $$y(\gamma_j) = x(\gamma_j), \; j = 1 \dots n$$
  $$y(\gamma) = 0 \quad \text{for } \gamma \in \Gamma \setminus \{\gamma_1, \dots, \gamma_n\}$$

  Then $y \in U \cap A$.

- A is sequentially compact:

  Let (x_n) be a sequence in A. For each $n \in \mathbb{N}$ let $J_n = \{ \gamma \in \Gamma, x_n(\gamma) \neq 0 \} \Rightarrow J_n$ is countable
  $J = \bigcup_n J_n$ is also countable

  Enumerate $J = \{ \gamma_k ; k \in \mathbb{N} \}$
  and perform an inductive construction:

$x_n^0 := x_n$

~~Let~~ given (x_n^{k-1}), let (x_n^k) be a subsequence of (x_n^{k-1}) s.t. $(x_n^k(\gamma_k))$ converges (in $[0,1]$)

Then take the diagonal sequence $z_n := x_n^n$.
Then (z_n) is a subsequence of (x_n) and $(z_n(\gamma))$ converges (in $[0,1]$) for $\gamma \in J$.
Moreover $z_n(\gamma) = 0$ for $\gamma \in \Gamma \setminus J$.

So, $z_n \to z$, where $z(\gamma) = \lim_n z_n(\gamma)$, $\gamma \in \Gamma$.

Since $z(\gamma) = 0$ for $\gamma \in \Gamma \setminus J$, $z \in A$. ┘

(4) Combination of (2) and (3): Let $\Gamma = \{0,1\}^{\mathbb{N}}$ and A be as in (3). Then A is sequentially compact, but $\overline{A} = [0,1]^\Gamma$ is not sequentially compact.

(5) Let Γ be uncountable, let A_0 be the set A from (3) and

$$A_1 = \{x \in [0,1]^\Gamma ;\ \{\gamma \in \Gamma ;\ x(\gamma) \neq 1\} \text{ is countable}\}$$

Then A_0, A_1 are two dense sequentially compact subsets of $[0,1]^\Gamma$, moreover $A_0 \cap A_1 = \emptyset$.

Fix a one-to-one convergent sequence (x_n) in A_1 [it exists, for example, due to sequential compactness of A_1]
Assume $x_n \to x$, $x \notin \{x_n, n \in \mathbb{N}\}$.

((x_n) may be also found explicitly: let (γ_n) be a one-to-one sequence in Γ, $x_n(\gamma) = 0$ if $\gamma = \gamma_k, k \le n$, $x_n(\gamma) = 1$ otherwise)

Let $X = A_0 \cup \{x_n, n \in \mathbb{N}\}$.

The A_0 is sequentially compact, A_0 is closed in X and X is not countably compact
(as the sequence (x_n) has no cluster point)

So: A_0 is sequentially compact, but $\overline{A_0}$ is not countably compact.

(6) Let Γ and $(f_n)_{n \in \mathbb{N}}$ be as in (2).
Let $K = \overline{\{f_n, n \in \mathbb{N}\}}$. Then K is a compact space which has no one-to-one convergent sequence.

Step 1: For $A \subset \mathbb{N}$ set $U(A) = \overline{\{f_n, n \in A\}}^K$

Then $U(A) \cap U(B) = \emptyset$ whenever $A \cap B = \emptyset$

Assume $A \cap B = \emptyset$. Let $\gamma \in \{0,1\}^{\mathbb{N}}$ be such that

$\gamma_n = 0$ for $n \in A$, $\gamma_n = 1$ for $n \in B$.

Then for $g \in U(A)$ we have $g(\gamma) = 0$

for $g \in U(B)$ we have $g(\gamma) = 1$.

So, $U(A) \cap U(B) = \emptyset$

Step 2: $U(A)$ is a clopen (closed and open) subset of K:

closed ... clear
open: $U(A) \cap U(\mathbb{N} \setminus A) = \emptyset$
$U(A) \cup U(\mathbb{N} \setminus A) = K$

Step 3: $U(A) \cap U(B) = U(A \cap B)$

$\ulcorner$ $K \setminus U(A\cap B) = U(\mathbb{N} \setminus (A \cap B)) = U((\mathbb{N}\setminus A) \cup (\mathbb{N} \setminus B))$

$= U(\mathbb{N}\setminus A) \cup U(\mathbb{N}\setminus B) = (K \setminus U(A)) \cup (K \setminus U(B)) =$

$= K \setminus (U(A) \cap U(B))$ $\lrcorner$

$\overline{C \cup D} = \overline{C} \cup \overline{D}$

Step 4: $U(A)$, $A \subset \mathbb{N}$ is a base of the topology of K

$\ulcorner$ $U(\mathbb{N}) = K$, by Step 3 we see that $U(A)$, $A \subset \mathbb{N}$ is a base of some topology on K.
By Step 2 this topology is weaker than the original topology.

Moreover, given $f, g \in K$, $f \neq g$, there is $p \in P = \{0,1\}^{\mathbb{N}}$ s.t. $f(p) \neq g(p)$

Assume $f(p) = 0$, $g(p) = 1$

Let $A = \{n \in \mathbb{N};\ p_n = 1\}$

Then $f \notin U(A)$, hence $f \in U(\mathbb{N} \setminus A)$
$g \notin U(\mathbb{N} \setminus A)$, hence $g \in U(A)$

By step 1 $U(A) \cap U(\mathbb{N} \setminus A) = \emptyset$.

Hence, the generated topology is Hausdorff.

So, it is a weaker Hausdorff topology on a compact space, hence it coincides with the original topology $\lrcorner$

Step 5: Assume that $(g_n)_{n\in\mathbb{N}}$ is a one-to-one sequence in K, $g_n \to g$. WLOG $g \notin \{g_n, n\in\mathbb{N}\}$.

Then there is a disjoint sequence $(A_n)_{n\in\mathbb{N}}$, $A_n \subset \mathbb{N}$ such that $g_n \in U(A_n)$

$$\begin{aligned} A &= \bigcup\{A_n, n \text{ odd}\} \\ B &= \bigcup\{A_n, n \text{ even}\} \end{aligned} \qquad \text{Then } U(A)\cap U(B)=\emptyset \text{ (by Step 1)}$$

but simultaneously $g_n \in U(A)$ for n odd
$g_n \in U(B)$ for n even

$\Rightarrow g \in U(A)\cap U(B)$. A contradiction $\perp$

(7) Let K be as in (6). Let $X = K\setminus\{f\}$ for some $f\in K$, $f \neq f_n, n\in\mathbb{N}$

Then X is not compact, as X is dense in K
X is not sequentially compact (by (2) or (6))

X is ctbly compact: Let (g_n) be a one-to-one sequence in X. Since K is compact, it has some cluster point in K. Since K contains no one-to-one convergent sequences, the sequence has at least two different cluster points. One of them differs from f, so (g_n) has a cluster point in $K\setminus\{f\} = X$.

(8) One more example:

- Family of infinite subsets of $\mathbb{N}$ is „almost disjoint" if any two distinct elements have finite intersection.

- Let $\mathcal{A}$ be a maximal almost disjoint family of subsets of $\mathbb{N}$ (infinite subsets). It exists by Zorn's lemma.

- Let $X = \mathbb{N} \cup \mathcal{A}$. We define a topology on X as follows:

  - points of $\mathbb{N}$ are isolated
  - $A \in \mathcal{A} \Rightarrow$ a neighborhood base of A is formed by

$$\{A\} \cup (A \setminus F), \quad F \subset \mathbb{N} \text{ finite.}$$

- Then X is Hausdorff and locally compact, thus completely regular. (It admits a one-point compactification)

- $\mathbb{N}$ is relatively sequentially compact in X

  [$(n_k, k \in \mathbb{N})$ a one-to-one sequence
  $A = \{n_k, k \in \mathbb{N}\}$ is infinite
  $\mathcal{A}$ maximal $\Rightarrow \exists\, B \in \mathcal{A}$: $A \cap B$ is infinite
  $A \cap B = \{k_\ell, \ell \in \mathbb{N}\}$, (k_ℓ) is a subsequence of (n_k). then $k_\ell \to B$ in X]

- There is no countably compact set Y with $\mathbb{N} \subset Y \subset X$

Γ First Assume $Y \subsetneqq X$. Fix $A \in X \setminus Y$.

Then $A = (x_k, k \in \mathbb{N})$, $x_k \to A$ in X

~~Then no subsequence of (x_k)~~ .

Then A is the unique cluster point of (x_k) in X, so (x_k) has no cluster point in Y.

Next assume $Y = X$:

Let (A_n) be an infinite one-to-one sequence in $\mathcal{A}$. Then $\{A_n, n \in \mathbb{N}\}$ is a closed discrete set in X (points of $\mathbb{N}$ are isolated in X, points of $\mathcal{A}$ are isolated in $\mathcal{A} = X \setminus \mathbb{N}$), so it has no cluster point in X ⌋