Remarks on Proposition 18 for non-closed operators:

Assume T is an one-to-one operator from X to Y which is not closed:

- Condition (i) fails:
  If $R(T) = Y$ and $T^{-1} \in L(Y,X)$, then T^{-1} is closed and hence T is also closed

- Condition (ii) may hold.
  In this case (i) and (iii) fail: (i) fails always (see above) and (iii) in this case reduces to (i)
  [By (ii) $R(T) = Y$. (iii) says that $R(T)$ is close and T^{-1} is continuous on $R(T)$. But we have even $R(T)=Y$, so T^{-1} is continuous on Y, i.e. $T^{-1} \in L(Y,X)$. This implies (i)]

EXAMPLE 1: $T: X \to X$ discontinuous linear bijection.
If $\dim X = \infty$, it exists ... take an algebraic basis $(x_i)_{i\in I}$ of X and define T by $x_i \mapsto \lambda_i x_i$, where $(\lambda_i)_{i\in I}$ is unbounded.
Then T has no closed extension [T is not closed and has no proper extension]

If T has a closed extension, then $\overline{T}$ is not one-to-one
[$x \in D(\overline{T}) \setminus D(T) \Rightarrow \overline{T}x \in Y = R(T) \Rightarrow \exists y \in D(T)$ $Ty = \overline{T}x$. hence $\overline{T}$ is not one-to-one.]

Example 2: $X = \ell_2$, $S: \ell_2 \to \ell_2$ $S(x_1, x_2, \dots) = (x_2, x_3, \dots)$
$\Rightarrow S \in L(X)$, S is not one-to-one, S is onto.
Let $\varphi \in X^{\#}$ be a discontinuous linear functional.
Set $D(T) = \{x \in \ell_2 ;\ x_1 = \varphi(Sx)\}$
$T = S\restriction_{D(T)}$

Then T is one-to one: $x \in D(T)$, $Tx = 0$
$\Rightarrow x_2 = x_3 = \dots = 0$, $x_1 = \varphi(Sx) = 0$ $\Big\} \Rightarrow x = 0$

T is onto: $x \in \ell_2$ $\quad y = (\varphi(x), x_1, x_2, \dots)$

$\Rightarrow Sy = x$, $y \in D(T)$, so $Ty = x$.

D(T) is dense: We know that $\ker \varphi$ is dense

$x \in \ell_2$, $\varepsilon > 0$

..... $\{z \in \ell_2;\ \varphi(z) = x_1\}$ is dense

so, find $z \in \ell_2$; $\varphi(z) = x_1$, $\|z - Sx\| < \varepsilon$

set $y = (x_1, z_1, z_2, \dots)$ Then $y \in D(T)$

$\|y - x\| = \|z - Sx\| < \varepsilon$

- Condition (iii) may hold.

In this case (i) and (ii) fail. (i) fails always (see above)

(ii) in this case reduces to (i)

T may have closed extension. In this case $\overline{T}$ satisfies (i) – (iii)

[R(T) dense, T^{-1} continuous on R(T) $\Rightarrow$ T^{-1} may be extended to $S \in L(Y, X)$. Moreover, $S = \overline{T^{-1}}$ (by construction of the extension). Hence, $G(S) = G(\overline{T^{-1}})$

So, if T has a closed extension, it must be S^{-1}.

Therefore, $\overline{T}$ is one-to-one and onto]

EXAMPLE 3 $\quad X = \ell_2$, $D(T) = c_{00}$ = finitely supported vectors

$$Tx = x, \quad x \in D(T)$$

T need not have a closed extension:

EXAMPLE 4: $X = \ell_2$, φ discontinuous linear functional

$$Tx = (\varphi(x), x_1, x_2, \dots)$$

T one-to-one

$D(T) = X$, T discontinuous $\Rightarrow$ T not closed, no proper extension

$\overline{R(T)} = X$, T^{-1} continuous as T is the inverse of the operator from EXAMPLE 2 above.