Proof of Theorem 35

(a) S symmetric. Then S is self-adjoint
$\iff C_S$ is a unitary operator

① WLOG S densely defined:

S self-adjoint $\Rightarrow$ S densely defined by definition

C_S unitary $\Rightarrow$
- $(I - C_S)$ one-to-one (Thm 32(a))
- $D(C_S) = H$
- $\{0\} = \ker(I - C_S) \overset{33(b)}{=} D(C_S) \cap (R(I - C_S))^{\perp} = (R(I-C_S))^{\perp}$

$\Rightarrow R(I - C_S)$ is dense
- $D(S) = R(I - C_S)$ by Thm 32 (b)

$\Rightarrow$ S densely defined

② Assume S is symmetric and densely defined. Then:

S self-adjoint

$\Updownarrow$ $\Downarrow$ Thm 30: S self-adjoint $\Rightarrow \sigma(S) \subset \mathbb{R}$
$\Uparrow$ Cor. 31, (iii) $\Rightarrow$ (i)

$i, -i \in \rho(S)$

$\Updownarrow$ $\Downarrow$ clear
$\Uparrow$ L29 $\Rightarrow$ $S + iI, S - iI$ one-to-one with continuous inverses

$R(S + iI) = R(S - iI) = H$

$\Updownarrow$ Thm 32(a)

$D(C_S) = R(C_S) = H$

$\Updownarrow$ Proposition 1

C_S unitary

$\lrcorner$

(b) U unitary, $I-U$ one-to-one $\Rightarrow$ $S = i(I+U)(I-U)^{-1}$
is self-adjoint and $C_S = U$

[Thm 34 $\Rightarrow$ S is symmetric and $C_S = U$

by (a) we deduce that S is self-adjoint]