

## Remarks on deficiency indices

①  $S, T$  symmetric operators  $\Rightarrow S \subset T \Leftrightarrow C_S \subset C_T$

⌈ This is easy ... use the formula for  $C_S$  and the inverse formula from Thm 32 (c) ⌋

②  $S$  ... a closed densely defined symmetric operator  
 deficiency indices:  $\text{codim } D(C_S) = \dim D(C_S)^\perp$   
 $\text{codim } R(C_S) = \dim R(C_S)^\perp$

(a)  $S$  is self-adjoint  $\Leftrightarrow$  both deficiency indices are 0

⌈  $S$  self-adjoint  $\stackrel{\text{Thm 35(a)}}{\Leftrightarrow} C_S \text{ unitary} \Leftrightarrow D(C_S) = R(C_S) = H$  ⌋

(b)  $S$  maximal symmetric  $\Leftrightarrow$  at least one of the deficiency indices is 0

⌈  $\Rightarrow$  : at least one of the indices are 0  $\Rightarrow D(C_S) = H$  or  $R(C_S) = H$

Assume  $T \supset S$  is symmetric. By ①  $C_T \supset C_S$ . But

$C_T$  is an isometry, so  $C_T = C_S$ , hence  $T = S$

$\Leftarrow$  : Assume both indices are  $> 0$ . Take  $x \in D(C_S)^\perp, y \in R(C_S)^\perp$

$\|x\| = \|y\| = 1$

$X := \text{span}(D(C_S) \cup \{x\}), Y := \text{span}(R(C_S) \cup \{y\})$

Define  $U: X \rightarrow Y$  by

$U(z + dx) = U(z) + dy, z \in D(C_S), d \in \mathbb{C}$

Then  $U$  is an isometry,  $D(U) = X, R(U) = Y, C_S \subsetneq U$

Thm 32 (b)

$S$  densely defined  $\Downarrow \Rightarrow R(I - C_S)$  is dense

$$\mathbb{I} - C_S \subset \mathbb{I} - U$$

$\Downarrow \Rightarrow R(I - U)$  is dense  $\Rightarrow I - U$  is one-to-one

L33(b)

Thm 34

$\Rightarrow U = C_T$  for a symmetric  $T$

Finally,  $C_S \not\subseteq U = C_T$ , hence  $S \not\subseteq T$  (by (1))

So,  $S$  is not maximal



(c)  $S$  admits a self-adjoint extension

$\Leftrightarrow$  the deficiency indices are equal

i.e., there is a linear isometry  $V: D(C_S)^\perp \xrightarrow{\text{onto}} R(C_S)^\perp$

$\Rightarrow$ :  $T \supset S$ ,  $T$  self-adjoint.

Then  $C_T \supset C_S$ ,  $C_T$  unitary.

$$\text{Hence } C_T(D(C_S)^\perp) = R(C_S)^\perp$$

$$\text{because } C_T(D(C_S)) = R(C_S)$$

and  $C_T$  is an isometry with  $D(C_T) = R(C_T) = H$

$\Leftarrow$ : Let  $V: D(C_S)^\perp \xrightarrow{\text{onto}} R(C_S)^\perp$  be a linear isometry

Then  $U(x+y) = C_S x + V y$ ,  $x \in D(C_S), y \in D(C_S)^\perp$   
is a unitary operator,  $C_S \subset U$

Moreover, as in (b) we see that  $\mathbb{I} - U$  is one-to-one,

hence  $U = C_T$  for a self-adjoint operator  $T$  (Thm 35(b))

Then  $S \subset T$



③ What happens if  $S$  is closed symmetric, but not <sup>necessarity</sup> densely defined  
 Deficiency indices make sense

(a) holds as well, by Thm 35 (a)  
 (b)  $\Leftrightarrow$  holds as well  
 (c)  $\Rightarrow$  holds as well } the same proof

Moreover:

•  $D(C_S) = H$  or  $R(C_S) = H \Rightarrow S$  densely defined

Assume  $D(C_S) = H$ . We know that  $I - C_S$  is one-to-one  
 (Thm 32 (b))

$$\Rightarrow \{0\} = \ker(I - C_S) = D(C_S) \cap R(I - C_S)^\perp = R(I - C_S)^\perp$$

$\uparrow$   $\underbrace{\hspace{2cm}}_{=H}$   
 (33 (b))

Hence  $R(I - C_S)$  is dense, so  $S$  is densely defined (Thm 32 (d))

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Assume  $R(C_S) = H$  and  $S$  is not densely defined

By Thm 32 (b)  $R(I - C_S)$  is not dense, so we may find  
 $x \in R(I - C_S)^\perp$ ,  $\|x\| = 1$

$$R(C_S) = H \Rightarrow \exists y \in D(C_S) : C_S y = x$$

$$\text{Then } 0 = \langle x, (I - C_S)y \rangle = \langle x, y \rangle - \langle x, x \rangle$$

$$\text{so, } \langle x, y \rangle = \langle x, x \rangle = 1$$

By the equality in Cauchy-Schwarz inequality we deduce  $y = x$

Hence  $C_S x = x$ ,  $I - C_S$  is not one-to-one.



## Questions:

- $IS \Rightarrow \alpha(S)$  valid without assuming  $S$  is densely defined?  
i.e.  $IS$  any maximal symmetric operator densely defined?
- $IS \Leftrightarrow \alpha(CS)$  valid without assuming  $S$  is densely defined?

This is related to the following questions:

- Assume  $U: D(U) \rightarrow R(U)$  is an isometry,  $D(U)$  closed,  $D(U) \neq H \neq R(U)$ ,  $I-U$  one-to-one.  
Is there an isometry  $V$  such that  $U \subseteq V$ ,  $I-V$  is one-to-one?
- Assume  $U$  is as above and, moreover, there is an isometry  $W: D(U)^\perp \xrightarrow{\text{onto}} R(U)^\perp$ .  
Is there a unitary operator  $V$  extending  $U$  such that  $I-V$  is one-to-one?