Lax-Milgram Lemma

Let H be a Hilbert space, $B : H \times H \to \mathbb{C}$ a mapping s.t.

- $x \mapsto B(x,y)$ is linear for $y \in H$
- $y \mapsto B(x,y)$ is conjugate linear for $x \in H$
- $\|B\| = \sup\{ |B(x,y)| \,;\, x,y \in B_H \} < \infty$

① Then $\forall x,y \in H : \; |B(x,y)| \leq \|B\| \; \|x\| \; \|y\|$

Γ $x = 0$ or $y = 0 \Rightarrow B(x,y) = 0$, so it is clear

$x \neq 0$ and $y \neq 0$ $\quad |B(x,y)| = \|x\| \cdot \|y\| \cdot \left| B\left(\frac{x}{\|x\|}, \frac{y}{\|y\|}\right) \right| \leq \|B\| \cdot \|x\| \cdot \|y\|$ $\lrcorner$

② Fix $x \in H$. Then $\varphi_x(y) = \overline{B(x,y)}$ is a linear functional on H, moreover by ① we have $\|\varphi_x\| \leq \|B\| \cdot \|x\|$

By Riesz theorem $\exists!\, Tx \in H$ s.t. $\quad \varphi_x(y) = \langle y, Tx \rangle$, $y \in H$.

and/moreover, $\|Tx\| = \|\varphi_x\| \leq \|B\| \cdot \|x\|$

Then $\quad B(x,y) = \overline{\varphi_x(y)} = \overline{\langle y, Tx \rangle} = \langle Tx, y \rangle$, $y \in H$

This shows, in particular, the uniqueness of T.

③ It remains to show that T is linear and $\|T\| = \|B\|$

Linearity: $\langle T(x_1 + x_2), y \rangle = B(x_1 + x_2, y) = B(x_1, y) + B(x_2, y) = \langle Tx_1, y \rangle + \langle Tx_2, y \rangle =$

$= \langle Tx_1 + Tx_2, y \rangle$

$\langle T(\alpha x), y \rangle = B(\alpha x, y) = \alpha B(x,y) = \alpha \langle Tx, y \rangle = \langle \alpha Tx, y \rangle$

So, by uniqueness we deduce $T(x_1 + x_2) = Tx_1 + Tx_2$, $T(\alpha x) = \alpha Tx$, so T is linear

Further, $\|T\| \leq \|B\|$ by ②

conversely (for $x,y \in B_H$) $|B(x,y)| = |\langle Tx, y \rangle| \leq \|Tx\| \cdot \|y\| \leq \|T\|$, so $\|B\| \leq \|T\|$