

PROPERTIES OF THE MEASURABLE CALCULUS

(14) $f \mapsto \tilde{f}(T)$ is a linear mapping $L^\infty(E_T) \rightarrow L(H)$

[obvious]

(15) $\tilde{f}(T) = \tilde{f}(T)^*$, $f \in L^\infty(E_T)$

[Let $x \in H$ be arbitrary. Then

$$\langle \tilde{f}(T)x, x \rangle = \int_{\sigma(T)} \bar{f} dE_{x,x}$$

$$\langle \tilde{f}(T)^*x, x \rangle = \langle x, \tilde{f}(T)x \rangle = \overline{\langle \tilde{f}(T)x, x \rangle} =$$

$$= \overline{\int_{\sigma(T)} f dE_{x,x}} = \int_{\sigma(T)} \bar{f} dE_{x,x} = \langle \tilde{f}(T)x, x \rangle$$

$E_{x,x} \geq 0$

So, $\tilde{f}(T)^* = \tilde{f}(T)$ by Proposition V.4 (c)]

(16) $\widetilde{fg}(T) = \tilde{f}(T)\tilde{g}(T)$, $f, g \in L^\infty(E_T)$

[we know it works if f, g are cts

• suppose g is cts.

Let $x, y \in H$

Let (f_n) be a uniformly bdd sequence of cts functions

s.t. $f_n \rightarrow f$ $|E_{g(T)x,y}| + |E_{x,y}|$ - a.e.

$$\text{Then } \langle \widetilde{fg}(T)x, y \rangle = \int_{\sigma(T)} f dE_{g(T)x,y} =$$

$$\stackrel{\text{①}}{=} \lim_{n \rightarrow \infty} \int_{\sigma(T)} f_n dE_{g(T)x,y} = \lim_{n \rightarrow \infty} \langle \tilde{f}_n(T)\tilde{g}(T)x, y \rangle =$$

the valid cts functions

$$\stackrel{\text{②}}{=} \lim_{n \rightarrow \infty} \langle \tilde{f}_n\tilde{g}(T)x, y \rangle = \lim_{n \rightarrow \infty} \int_{\sigma(T)} f_n\tilde{g} dE_{x,y} \stackrel{\text{③}}{=} \int_{\sigma(T)} f\tilde{g} dE_{x,y} = \langle \widetilde{fg}(T)x, y \rangle$$

① = Lebesgue dom. conv. thm

• $f, g \in L^\infty(E_T)$ general

Let $x, y \in H$

Let (g_n) be a unif. bdd sequence of cts functions s.t.

$$g_n \rightarrow g \quad |E_{+10}| + |E_{+1\tilde{f}(T)^*y}| - a.e.$$

$$\langle \tilde{f}(T) \tilde{g}(T) x, y \rangle = \langle \tilde{g}(T) x, \tilde{f}(T)^* y \rangle = \int_{\sigma(T)} g \, dE_{+1\tilde{f}(T)^*y} =$$

Lebesgue dom. conv. thm

$$\downarrow = \lim_{n \rightarrow \infty} \int_{\sigma(T)} g_n \, dE_{+1\tilde{f}(T)^*y} = \lim_{n \rightarrow \infty} \langle \tilde{g}_n(T) x, \tilde{f}(T)^* y \rangle =$$

$$= \lim_{n \rightarrow \infty} \langle \tilde{f}(T) \tilde{g}_n(T) x, y \rangle = \lim_{n \rightarrow \infty} \langle \tilde{f} \tilde{g}_n(T) x, y \rangle =$$

the previous case

$$= \lim_{n \rightarrow \infty} \int_{\sigma(T)} f g_n \, dE_{+10} = \int_{\sigma(T)} f g \, dE_{+10} = \langle f \tilde{f}(T) x, y \rangle$$

Lebesgue dom. conv. thm.

(17) Summarizing (14), (15), (16) : $f \mapsto \tilde{f}(T)$ is a *-homomorphism $L^\infty(E_T) \rightarrow L(H)$

In part : f real-valued (except on a set for \mathcal{N}) $\Rightarrow \tilde{f}(T)$ self-adjoint

(18) $f \geq 0 \Rightarrow \left(\tilde{f}(T) \geq 0, \text{ moreover } \tilde{f}(T) = 0 \Leftrightarrow f = 0 \right)$

• $f \geq 0 \Rightarrow \langle \tilde{f}(T) x, x \rangle = \int f \, dE_{+x} \geq 0$ as $E_{+x} \geq 0$

$$\tilde{f}(T) \geq 0 \Rightarrow \forall x \langle \tilde{f}(T) x, x \rangle = 0 \Rightarrow \forall x \int f \, dE_{+x} = 0$$

As $f \geq 0$, we deduce $f = 0$ E_{+x} -a.e.

Hence $f = 0$ except on a set for \mathcal{N}

(19) $\widehat{f}(T) = 0 \Rightarrow f = 0$ except on a set from \mathcal{N}

$f \geq 0 \Rightarrow$ by (18)

f real-valued $\Rightarrow f = f^+ - f^-$, $f^+, f^- \in L^\infty(E_T)$

Then $\widehat{f}(T) = \widehat{f}^+(T) - \widehat{f}^-(T)$, thus $\widehat{f}^+(T) = \widehat{f}^-(T)$.

So, $(\widehat{f}^+)^2(T) \stackrel{(16)}{=} \widehat{f}^+(T) \widehat{f}^+(T) = \widehat{f}^+(T) \widehat{f}^-(T) =$

$\stackrel{(16)}{=} \widehat{f^+ f^-}(T) = \widehat{0}(T) = 0$

Thus $f^+ = 0$ except on a set from \mathcal{N} and the same holds for f^- , thus $f = 0$ except on a set from \mathcal{N}

f complex $\Rightarrow \widehat{f}(T) = \widehat{\operatorname{Re} f}(T) + c \widehat{\operatorname{Im} f}(T)$

$\Rightarrow \widehat{\operatorname{Re} f}(T) = 0 \ \& \ \widehat{\operatorname{Im} f}(T) = 0$

$\Rightarrow \operatorname{Re} f = 0, \operatorname{Im} f = 0$ except on a set from \mathcal{N}
 $\Rightarrow f = 0$ except on a set from \mathcal{N} .

(20) So, $f \mapsto \widehat{f}(T)$ is a $*$ -isomorphism, so it is an isometry.

- In particular:
- $\widehat{f}(T)$ is always a normal operator
 - $\widehat{f}(T)$ is self-adjoint $\Leftrightarrow f$ real valued (except on a set from \mathcal{N})
 - $\sigma(\widehat{f}(T)) = \sigma(f) = \operatorname{ess\,rang}(f)$

$= \{ \lambda \in \mathbb{C}; \forall \epsilon > 0 \ f^{-1}(U(\lambda, \epsilon)) \notin \mathcal{N} \}$

(21) $(f_n) \subset L^\infty(E_T)$, $f_n \rightarrow f$ pointwise except on a set from \mathcal{N}
 (f_n) unif. bdd

$$\Rightarrow f \in L^\infty(E_T), \quad \langle \tilde{f}_n(T)_{x,y} \rangle \rightarrow \langle \tilde{f}(T)_{x,y} \rangle$$

[More definitions and Lebesgue dominated conv. thm]

(22) $f \in L^\infty(E_T)$, $g \in \mathcal{L}(\sigma(\tilde{f}(T))) = \mathcal{L}(\overline{\text{ran}} f)$

$$\Rightarrow \widehat{g \circ f}(T) = \widehat{g}(\tilde{f}(T))$$

$$\Gamma A := \{ g \in \mathcal{L}(\sigma(\tilde{f}(T))) ; \widehat{g \circ f}(T) = \widehat{g}(\tilde{f}(T)) \}$$

- A is linear
- $1 \in A$, $cd \in A$
- $g \in A \Rightarrow \overline{g} \in A$ (as $\overline{g \circ f} = \overline{g} \circ f$)
- $g_1, g_2 \in A \Rightarrow g_1 g_2 \in A$
 $\Gamma \widehat{(g_1 g_2) \circ f}(T) = \widehat{(g_1 \circ f) \cdot (g_2 \circ f)}(T) = \widehat{g_1 \circ f}(T) \widehat{g_2 \circ f}(T)$
 $= \widehat{g_1}(\tilde{f}(T)) \widehat{g_2}(\tilde{f}(T)) = \widehat{g_1 g_2}(\tilde{f}(T)) \quad \square$

• A is norm-closed

So, by Stone-Weierstrass thm we get $A = \mathcal{L}(\sigma(\tilde{f}(T)))$

$$(23) ST = TS \Rightarrow S \widehat{f}(T) = \widehat{f}(T) S$$

Γ we know it works if f is cts

• f general: fix $x, y \in H$. Find (f_n) unif. bdd sequence of cts fcts
 s.t. $f_n \rightarrow f = |E_{Sx,y}| + |E_{x,Sy}| - q.l.$

$$\begin{aligned} \text{Then } \langle S \widehat{f}(T)_{x,y} \rangle &= \langle \widehat{f}(T)_{x,Sy} \rangle = \int_S dE_{x,Sy} = \lim_n \int f_n dE_{x,Sy} \\ &= \lim_n \langle \widehat{f_n}(T)_{x,Sy} \rangle = \lim_n \langle S \widehat{f_n}(T)_{x,y} \rangle = \lim_n \langle \widehat{f_n}(T)_{Sx,y} \rangle = \\ &= \lim_n \int f_n dE_{Sx,y} = \int f dE_{Sx,y} = \langle \widehat{f}(T)_{Sx,y} \rangle \quad \square \end{aligned}$$