

Proof of Theorem VI.12

(a) $\Phi(f) + \Phi(g) \subset \Phi(f+g)$

$$\Gamma x \in D(\Phi(f) + \Phi(g)) = D(\Phi(f)) \cap D(\Phi(g))$$

$$\Rightarrow f, g \in L^2(E_{T,x}) \Rightarrow f+g \in L^2(E_{T,x}) \Rightarrow x \in D(\Phi(f+g))$$

Moreover, for $x, y \in D(\Phi(f) + \Phi(g))$ we have

$$\langle \Phi(f) + \Phi(g), x, y \rangle = \langle \Phi(f), x, y \rangle + \langle \Phi(g), x, y \rangle =$$

$$= \int f dE_{T,x,y} + \int g dE_{T,x,y} = \int (f+g) dE_{T,x,y} = \langle \Phi(f+g), x, y \rangle$$

Since $D(\Phi(f)) \cap D(\Phi(g)) = D(\Phi(|f|+|g|))$ is dense,

we deduce $\Phi(f) + \Phi(g) = \Phi(f+g)$ \square

(b) $\Phi(f)\Phi(g) \subset \Phi(fg)$ & $D(\Phi(f)\Phi(g)) = D(\Phi(g)) \cap D(\Phi(fg))$

$$\Gamma x \in D(\Phi(f)\Phi(g)) \Leftrightarrow x \in D(\Phi(g)) \text{ & } \Phi(g)x \in D(\Phi(f))$$

$$\Leftrightarrow g \in L^2(E_{T,x}) \text{ & } f \in L^2(E_{\Phi(g)x, \Phi(g)x}) \Leftrightarrow (*)$$

Compute $E_{\Phi(g)x, \Phi(g)x}(A) = \langle E(A)\Phi(g)x, \Phi(g)x \rangle =$

$$= \|E(A)\Phi(g)x\|^2 = \lim_{n \rightarrow \infty} \|E(A)\Phi(g_n)x\|^2 =$$

\uparrow
E(A) is an OS projection

\uparrow
 g_n as in proof of Thm 11.15

$$= \lim_{n \rightarrow \infty} \|\Phi \circ (\chi_A g_n)\|^2 = \lim_{n \rightarrow \infty} \int_A |g_n|^2 dE_{T,x} = \int_A |g|^2 dE_{T,x}$$

\uparrow
Thm 8.11

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So, $E_{\Phi(g)_+, \Phi(g)_+}$ is absolutely continuous with respect to $E_{+,+}$,
with density $|g|^2$

$$\text{So, } \int |f|^2 dE_{\Phi(g)_+, \Phi(g)_+} = \int |fg|^2 dE_{+,+}$$

Here, continue:

$$(*) \Leftrightarrow g \in L^2(E_{+,+}) \Leftrightarrow fg \in L^2(E_{+,+}) \Leftrightarrow x \in D(\Phi(g)) \cap D(\Phi(fg))$$

The equality $\Phi(fg)_+ = \Phi(f)_+ \Phi(g)_+$:

• f, g bdd ... by Thm 8.9(a)

• f bdd, g not necessarily ... take g_n as in proof of Thm 8.9(b)

$$\text{Thm } \Phi(g)_+ = \lim_{n \rightarrow \infty} \Phi_0(g_n)_+ \quad , + \infty \in D(\Phi(g))$$

f bdd $\Rightarrow \Phi(f) \in L(H)$, so

$$\Rightarrow \Phi(f) \Phi(g)_+ = \lim_{n \rightarrow \infty} \Phi(f) \Phi_0(g_n)_+ =$$

$$= \lim_{n \rightarrow \infty} \Phi_0(fg_n)_+ = \Phi(fg)_+$$

$$\left(\begin{array}{c} \uparrow (*) \\ fg_n \rightarrow fg \text{ in } L^2(E_{+,+}) \end{array} \right)$$

• f, g general ... take f_n as in proof of Thm 8.9(b)

$$\begin{aligned} \Rightarrow \Phi(f) \Phi(g)_+ &= \lim_{n \rightarrow \infty} \Phi_0(f_n) \Phi(g)_+ = \lim_{n \rightarrow \infty} \Phi_0(f_n g)_+ \stackrel{(*)}{=} \\ &= \Phi(fg)_+ \end{aligned}$$

In (*) we use Lebesgue dominated convergence theorem

$$\langle \Phi_0(t g_n) + i0 \rangle = \int f g_n dE_{x,y} \rightarrow \int f g dE_{x,y} = \langle \Phi(tg) + i0 \rangle$$

$|fg|$
 is an integrable
 majorant.

$$(c) \quad \underline{\Phi}(t)^* = \underline{\Phi}(\bar{t}) \quad , \quad \underline{\Phi}(t) \underline{\Phi}(t)^* = \underline{\Phi}(t)^* \underline{\Phi}(t) = \underline{\Phi}(|t|^2)$$

$$\Gamma \bullet D(\underline{\Phi}(\bar{t})) = D(\underline{\Phi}(t)) \quad \dots \text{clear, as } |\bar{t}|^2 = |t|^2$$

$$\bullet x, y \in D(\underline{\Phi}(t)) \Rightarrow$$

$$\begin{aligned} \langle \underline{\Phi}(t)x, y \rangle &= \lim_n \langle \Phi_0(t_n) + i0 \rangle = \lim_n \langle +i \Phi_0(t_n)^* g \rangle \\ &= \lim_n \langle +i \Phi_0(\bar{t}_n) g \rangle = \langle +i \Phi(\bar{t}) g \rangle \end{aligned}$$

$$\text{So, } \underline{\Phi}(\bar{t}) \subset \underline{\Phi}(t)^*$$

Conversely. Let $A_n := \{ \lambda; |\lambda| \leq n \}$. Then

$$\begin{aligned} E(A_n) \underline{\Phi}(\bar{t})^* &\subset (\underline{\Phi}(t) E(A_n))^* = (\underline{\Phi}(t) \Phi_0(\psi_{A_n}))^* \\ &\stackrel{(b)}{=} (\underline{\Phi}(t \psi_{A_n}))^* = \underline{\Phi}(\bar{t} \psi_{A_n}) \end{aligned}$$

$$x \in D(\underline{\Phi}(t)^*) \Rightarrow E(A_n) \underline{\Phi}(t)^* x \rightarrow \underline{\Phi}(t)^* x$$

$$\Rightarrow \underline{\Phi}(t)^* x = \lim_{n \rightarrow \infty} \underline{\Phi}(\bar{t} \psi_{A_n}) x$$

$$\Rightarrow \langle \Phi(f)^*_{t,t} \rangle = \lim_{n \rightarrow \infty} \langle \Phi(f \chi_{A_n})_{t,t} \rangle = \lim_{n \rightarrow \infty} \int |f \chi_{A_n}|^2 dE_{t,t} = \int |f|^2 dE_{t,t}, \text{ so } t \in D(\Phi(t)).$$

Hence $D(\Phi(f)^*) \subset D(\Phi(f))$. (It follows that $\Phi(t)^* = \Phi(f)$)

$$\bullet \Phi(t)^* \Phi(t) = \Phi(t) \Phi(t)^* \subset \Phi(|f|^2) \quad \text{by (b)}$$

$$\Gamma D(\Phi(t)^* \Phi(t)) = D(\Phi(t)) \cap D(\Phi(|f|^2))$$

$$D(\Phi(f) \Phi(t)^*) = D(\Phi(f)) \cap D(\Phi(|f|^2)) \quad \text{J}$$

$$\bullet D(\Phi(|f|^2)) \subset D(\Phi(t))$$

$$\Gamma x \in D(\Phi(|f|^2)) \Rightarrow |f|^2 \in L^2(E_{x,x}) \Rightarrow |f|^2 \in L^1(E_{x,x}) \Rightarrow f \in L^2(E_{x,x}), \text{ so } x \in D(\Phi(t)) \quad \text{J}$$

(d) $\Phi(t)$ is a closed operator

$$\Gamma \Phi(t) = \Phi(f)^* \quad \text{J}$$

(e) $\Phi(t)$ is cts $\Leftrightarrow f$ is essentially s.c.c.d

$\Gamma \Leftarrow$: by Thm 8

\Rightarrow : $\Phi(t)$ s.c.c.d $\Rightarrow D(\Phi(t)) = H$, f_n as above

(i.e. $f_n = f \chi_{A_n}$, $A_n = \{x, |f(x)| \leq n\}$)

$$\Rightarrow \Phi_0(f_n)x \rightarrow \Phi(t)x, \quad x \in H$$

uniform boundedness principle $\Rightarrow (\Phi_0(f_n))$ is uniformly s.c.c.d

but $\|\Phi_0(f_n)\| = \|f_n\|_\infty \Rightarrow f$ ess. s.c.c.d $\quad \text{J}$