

Proposition VI.13

$$T = \int f dE \Rightarrow \sigma(T) = \text{ess-rng}(f);$$

$$\lambda \in \sigma \Rightarrow \ker(\lambda I - T) = R(E(f^{-1}(\{\lambda\})))$$

so,  $\lambda \in \sigma$  is an eigenvalue of  $T \Leftrightarrow E(f^{-1}(\lambda)) \neq 0$

Proof: ①  $\lambda I - \Phi(f) = \Phi(\lambda - f)$ , so look just at  $\Phi(f)$

$$\textcircled{2} A := f^{-1}(\{0\}) \Rightarrow R(E(A)) = \ker \Phi(f)$$

$$\Gamma \subset: \Phi(f) \upharpoonright R(E(A)) = 0, \text{ as } f=0 \text{ on } A$$

$$\Phi(f) E(A) = \Phi(f) \Phi_0(\chi_A) = \Phi(f \chi_A) \stackrel{\downarrow}{=} \Phi(0) = 0$$

Thm 29(b)

$$D(\Phi(f) \Phi_0(\chi_A)) = D(\Phi_0(\chi_A)) \cap D(\Phi(f \chi_A)) = D(\Phi_0(\chi_A)) \cap D(\Phi(f \chi_A)) = D(\Phi(f \chi_A))$$

$$\triangleright: \text{Let } g(x) = \begin{cases} 0 & x \in A \\ \frac{1}{f(x)} & x \notin A \end{cases}$$

$\Rightarrow g$  is  $A$ -measurable,  $g \cdot f = \chi_{A^c}$

$$\text{by Thm 11(b): } \Phi(g) \Phi(f) \subset \Phi(\chi_{A^c}) = E(A^c)$$

$$\text{So, if } x \in \ker \Phi(f) \Rightarrow \Phi(f)x = 0 \Rightarrow \Phi(g) \Phi(f)x = 0 \Rightarrow E(A^c)x = 0 \Rightarrow x = E(A)x \Rightarrow x \in R(E(A))$$

③ By ② we see:  $0$  is an eigenvalue  $\Leftrightarrow E(A) \neq 0$

④ Suppose  $E(A) = 0$ . Then  $\frac{1}{f}$  is  $A$ -measurable

$$\Phi\left(\frac{1}{f}\right) \Phi(f) \subset \Phi(1) = I \quad (\text{Thm 12})$$

$$D(\Phi\left(\frac{1}{f}\right) \Phi(f)) = D(\Phi(f)) \quad (\text{Thm 12(b)}) \Rightarrow \Phi\left(\frac{1}{f}\right) = \Phi(f)^{-1}$$

$$0 \notin \sigma(\Phi(f)) \Leftrightarrow \Phi\left(\frac{1}{f}\right) \in L(H) \Leftrightarrow \frac{1}{f} \text{ ess. bdd} \quad (\text{Thm 12(e)})$$

$$\Leftrightarrow 0 \notin \text{ess-rng}(f)$$