LEMMA VI.9 $\quad T = \int g \, dE \quad$ (g bdd)

$\Rightarrow$ the spectral measure of T is

$$E_T(A) = E(g^{-1}(A)), \quad A \in \mathcal{A}_T = \{A \subset \mathbb{C} ; g^{-1}(A) \in \mathcal{A}\}$$

Proof: Thm 8 (e) $\Rightarrow$ $\forall f \in \mathcal{C}(\sigma(T))$:

$$\tilde{f}(T) = \int f \circ g \, dE$$

So, for $x, y \in H$ and $f \in \mathcal{C}(\sigma(T))$:

$$\langle \tilde{f}(T) x, y \rangle = \langle \left(\int (f \circ g) \, dE \right) x, y \rangle = \int (f \circ g) \, dE_{x,y} =$$

$$= \int f \, d g(E_{x,y})$$

So, $(E_T)_{x,y} = g(E_{x,y})$, i.e.

$$(E_T)_{x,y}(A) = E_{x,y}(g^{-1}(A)).$$

This shows that $\mathcal{A}_T$ is as above. Moreover,

$$\langle E_T(A) x, y \rangle = \langle \widetilde{\chi_A}(T) x, y \rangle = \int \chi_A \, d g(E_{x,y}) =$$

$$= \int \chi_A \circ g \, dE_{x,y} = \int \chi_{g^{-1}(A)} \, dE_{x,y} = E_{x,y}(g^{-1}(A)) =$$

$$= \langle E(g^{-1}(A)) x, y \rangle$$

Corollary VI.10 $T \in \mathcal{L}(H)$ bdd normal

$\Rightarrow$ $T = \int id \, d\tilde{E}_T$. Moreover $T = \int id \, dE \Rightarrow E = E_T$

Γ $\langle (\int id \, d E_T) x, y \rangle = \int id \; d(E_T)_{x,y} =$

$= \langle \widetilde{id}(T) x, y \rangle = \langle T x, y \rangle$

The uniqueness follows from Lemma 9 $\rfloor$