**Lemma VI.15** $T$ self-adjoint (unbdd), $C_T$ its Cayley transform, $E$ ... the spectral measure of $C_T$

Then $T = \int i \frac{1+z}{1-z} \, dE(z)$

Proof ① $C_T$ is unitary $\Rightarrow \sigma(C_T) \subset \mathbb{T} = \{\lambda \in \mathbb{C}, |\lambda| = 1\}$

② $I - C_T$ is one-to-one, i.e. $1$ is not an eigenvalue of $C_T$ $\Rightarrow E(\{1\}) = 0$ [by Cor. 10 & Prop. 13]

③ $f(z) = i \cdot \frac{1+z}{1-z}$ is $\mathcal{A}$-measurable (defined $E$-a.e.)

$f$ is real-valued (essentially) ... $i \frac{1+z}{1-z} = i \frac{(1+z)(1-\bar{z})}{(1+z)(1-\bar{z})} =$

$= i \frac{1+z-\bar{z}-|z|^2}{|1-z|^2} \underset{|z|=1,\, z \in \mathbb{T}}{=} - \frac{2 \operatorname{Im} z}{|1-z|^2}$

$\Rightarrow S := \int f \, dE$ is self-adjoint (Thm 12 (c))

④ Moreover: $f(z) \cdot (1-z) = i(1+z)$

So, by Thm 12 (b): $S(I - C_T) = i(I + C_T)$

$$\Rightarrow \underbrace{S(I - C_T)(I - C_T)^{-1}}_{I \restriction D(T)} = \underbrace{i(I + C_T)(I - C_T)^{-1}}_{T \ (\text{Thm V.3 (c)})}$$

$\left( \begin{array}{l} R(I - C_T) = D(T) \\ \text{Thm V.3 (b)} \end{array} \right)$

$\Rightarrow S \restriction D(T) = T \Rightarrow T \subset S$

$S, T$ selfadjoint $\Rightarrow T = S$

Lemma V.16 $F$ abstract spectral measure on $\mathcal{A}$

$\varphi: \sigma \to \mathbb{C}$ $\mathcal{A}$-measurable

$E(A) = F(\varphi^{-1}(A))$ ; $A \in \mathcal{A}' = \{A \subset \mathbb{C},\ \varphi^{-1}(A) \in \mathcal{A}\}$

(1) $E$ is an abstract spectral measure

- properties (i) – (vi) are obvious
- (vii) : $E_{x,y}$ is a complex Borel measure for each $x, y \in H$

By remark after 2.6 it is enough to prove it for $E_{x,x}$, $x \in H$

Note that $E_{x,x} = \varphi(F_{x,x})$. To simplify notation set $\mu := F_{x,x}$, $\nu := E_{x,x} = \varphi(\mu)$

Let $A \in \mathcal{A}'$ set $\beta := \sup\{\nu(B);\ B \subset A \text{ Borel}\}$

$\gamma := \inf\{\nu(C);\ C \supset A \text{ Borel}\}$

Let $B_n \subset A \subset C_n$ be Borel s.t. $\nu(B_n) > \beta - \frac{1}{n}$

$\nu(C_n) < \gamma + \frac{1}{n}$

$B := \bigcup_n B_n$, $C := \bigcap_n C_n$ $\Rightarrow B, C$ Borel, $B \subset A \subset C$

$\nu(B) = \beta$, $\nu(C) = \gamma$

We will be done if we prove $\nu(C \setminus B) = 0$ (i.e. $\beta = \gamma$)

Suppose $\nu(C \setminus B) > 0$. Then either $\nu(C \setminus A) > 0$ or $\nu(A \setminus B) > 0$

Suppose $\nu(C \setminus A) > 0$ (the other case is analogous)

Then $\mu(\varphi^{-1}(C \setminus A)) = \nu(C \setminus A) > 0$, $\varphi^{-1}(C \setminus A) \in \mathcal{A}$

Since $\mu$ is a Borel measure, there is $D \subset \varphi^{-1}(C \setminus A)$ Borel s.t. $\mu(D) > 0$ [in fact $\mu(D) = \mu(\varphi^{-1}(C \setminus A))$]

Since Borel measures on $\mathbb{C}$ are regular, there is $D_1 \subset D$ compact s.t. $\mu(D_1) > 0$

By Luzin's theorem there is $K \subset D_1$ compact s.t. $\mu(K) > 0$ & $\varphi \restriction_K$ is continuous

Then $\varphi(K) \subset C \setminus A$, $\varphi(K)$ is compact, hence Borel, and $\nu(\varphi(K)) = \mu(\varphi^{-1}(\varphi(K))) \geq \mu(K) > 0$

So, $C \setminus \varphi(K) \supset A$ is a Borel set with $\nu(C \setminus \varphi(K)) < \gamma$, a contradiction completing the proof.

(2) $f : \mathbb{C} \to \mathbb{C}$ $\mathcal{A}'$-measurable $\Rightarrow \int f\,dE = \int (f \circ \varphi)\,dF$

- $\int |f|^2\,dE_{x,x} = \int |f|^2\,d\varphi(F_{x,x}) = \int |f\circ\varphi|^2\,dF_{x,x}$

  $\Rightarrow$ the two domains coincide

- $x, y \in D\left(\int f\,dE\right) \Rightarrow$

  $\langle (\int f\,dE)x, y\rangle = \int f\,dE_{x,y} = \int f\,d\varphi(F_{x,y}) = \int f \circ F\,dF_{x,y}$

  $= \langle (\int (f\circ\varphi)\,dF)x, y\rangle$

Theorem VI.17 $T$ Self-adjoint $\Rightarrow \exists!$ abstract spectral measure $E$ s.t. $T = \int id\, dE$

Moreover, this $E$ is the image of the spectral measure of $C_T$ under $z \mapsto i\frac{1+z}{1-z}$

Proof ① Let $F$ be the spectral measure of $C_T$
$\varphi(z) = i\frac{1+z}{1-z}$, $z \in \mathbb{C}\setminus\{1\}$

Since $F(\{1\}) = 0$, $\varphi$ is measurable (see the proof of Lemma 15)

Let $E = \varphi(F)$. By L 16 $E$ is an abstract spectral measure and
$$\int id\, dE = \int id\circ\varphi\, dF = \int \varphi\, dF \underset{L:15}{=} T$$

② Uniqueness: Let $E$ be an abstract spectral measure such that $T = \int id\, dE$

$\sigma(T) \subset \mathbb{R} \Rightarrow$ ess-rng$(id) \subset \mathbb{R}$,
Set $g(z) = \frac{z-i}{z+i}$. for $z \in \mathbb{R}$ we have $|g(z)| = 1$,
$\frac{1}{g(z)} = \overline{g(z)}$

$\Rightarrow U := \int g\, dE$ is a unitary operator

By Prop. 14 $E_U = g(E)$

Further, $\varphi\circ g = id$ $E$-a.e. $\Rightarrow$

$$\Rightarrow T = \int id\, dE = \int \varphi\circ g\, dE = \int \varphi\, dE_U$$

$$\Rightarrow T(I-U) = \left(\int\varphi\, dE_U\right)(I-U) = \left(\int\varphi\, dE_U\right)\left(\int(1-z)\, dE_U(z)\right)$$

$$\overset{\text{Thm 12 (3)}}{=} \int i(1+z)\, dE_U(z) = i(I+U) \Rightarrow U = C_T$$

Since $E = \varphi(E_U)$, we deduce the uniqueness.

Corollary VI.18 $T$ selfadjoint $\Rightarrow$ ($T$ bdd $\Leftrightarrow$ $\sigma(T)$ bdd)

Proof $\Rightarrow$ clear, spectrum of a bdd operator is compact

$\Leftarrow$ $T = \int id \, dE$, $E(\mathbb{C} \setminus \sigma(T)) = 0$
$\sigma(T)$ bdd $\Rightarrow$ $id$ is essentially bdd

$\Rightarrow$ $T$ bdd