# Proof of Lemma VI.19

Let $T$ be a closely defined closed operator

(1) $I + T^*T$ is one-to-one

$\ulcorner$ Let $x \in D(I+T^*T) = D(T^*T)$ be such that $(I+T^*T)x = 0$

Then $0 = \langle (I+T^*T)x, x\rangle = \langle x, x\rangle + \langle T^*Tx, x\rangle =$

$\overset{x \in D(T),\ Tx \in D(T^*)}{=} \langle x, x\rangle + \langle Tx, Tx\rangle \geq \langle x, x\rangle \implies x = 0$ $\lrcorner$

(2) Recall that $G(T^*) = V(G(T)^\perp)$, where $V(x,y) = (-y, x)$ (by Lemma V.24)

So, $G(T)^\perp = V(G(T^*))$

Let $P$ be the OG projection of $H \oplus H$ onto $G(T)$
Define $B, C \in L(H)$ by

$$Bu = \pi_1 P(u, 0) \qquad u \in H \qquad \left(\text{where } \begin{array}{l} \pi_1(x,y) = x \\ \pi_2(x,y) = y \end{array}\right)$$
$$Cu = -\pi_2 (I-P)(u, 0)$$

Clearly $B, C \in L(H)$, $\|B\| \leq 1$, $\|C\| \leq 1$

Further, $P(u,0) = (Bu, TBu)$ $\quad (R(P) = G(T))$
$(I-P)(u,0) = (T^*Cu, -Cu)$
(as $R(I-P) = \operatorname{Ker} P = G(T)^\perp =$
$= V(G(T^*))$ )

So, $(u,0) = P(u,0) + (I-P)(u,0) = (Bu + T^*Cu, TBu - Cu)$
for $u \in H$

It follows $TBu - Cu = 0$, so $-C = TB$

$Bu + T^*Cu = u$, so $I = B + T^*C = B + T^*TB$
$= (I + T^*T)B$

So, $I = (I+T^*T)B$. Hence, $R(I+T^*T) = H$.
As $I+T^*T$ is one-to-one (by ①),
$R(B) = D(I+T^*T) = D(T^*T)$

③ Conclusion from ① and ②: $I+T^*T$ is one-to-one and onto,
$B = (I+T^*T)^{-1}$, $C = TB$, $\|B\| \le 1$, $\|C\| \le 1$

④ $B \geq 0$ [Hence, (a) and (b) hold]

[The computation in ① yields $\langle (I+T^*T)t, t\rangle \geq 0$, $t \in D(T^*T)$

So, for $u \in H$: $\langle Bu, u\rangle = \langle Bu, (I+T^*T)Bu\rangle \geq 0$ ]

⑤ $D(T^*T)$ is dense in $H$

[$D(T^*T) = R(B)$, $B \geq 0$, so $B$ is selfadjoint.
$B$ is one-to-one (being an inverse), so $R(B)$ is dense
(by Prop. V.23) ]

⑥ $T^*T$ is self-adjoint

[ $(I+T^*T) = B^{-1}$ $\Rightarrow$ $I+T^*T$ is self-adjoint by Prop. V.28(e)
so $T^*T$ is self-adjoint as well (Prop. V.22(b)) ]

⑦ $T = \overline{T\restriction_{D(T^*T)}}$

[i.e., $G(T\restriction D(T^*T))$ is dense in $G(T)$. If not, then $\exists\, (t, Tt) \in G(T)$
$(t, Tt) \perp G(T\restriction D(T^*T)) = G(T\restriction R(B))$ $\Rightarrow$ $\forall u \in H: (t, Tt) \perp (Bu, TBu)$
So, $0 = \langle t, Bu\rangle + \langle Tt, TBu\rangle = \langle t, Bu\rangle + \langle t, T^*TBu\rangle = \langle t, Bu + T^*TBu\rangle$

$Bu \in D(T^*T) \Rightarrow TBu \in D(T^*)$ $\qquad$ $= \langle t, u\rangle$

So, $t = 0$ ]