Proof of Lemma VI.20 Let T be a normal operator on H, i.e. $T^*T = TT^*$

(1) $\|Tx\| = \|T^*x\|$, $x \in D(T^*T) = D(TT^*)$

Γ $T^*T = TT^* \Rightarrow D(T^*T) = D(TT^*)$

$x \in D(T^*T) \Rightarrow \langle T^*Tx, x\rangle = \langle TT^*x, x\rangle$

$\langle Tx, Tx\rangle$ $\quad$ $\langle T^*x, T^*x\rangle$

$x \in D(T), Tx \in D(T^*)$ $\quad$ $x \in D(T^*), T^*x \in D(T)$ $\lrcorner$

(2) $D(T) \subset D(T^*)$, $\|T^*x\| \leq \|Tx\|$, $x \in D(T)$

Γ $x \in D(T) \overset{L19(c)}{\Rightarrow} \exists (x_n) \subset D(T^*T)$: $x_n \to x$ & $Tx_n \to Tx$

$y \in D(T) \Rightarrow \langle Ty, x\rangle = \lim_n \langle Ty, x_n\rangle = \lim_n \langle y, T^*x_n\rangle$

$x_n \in D(T^*T) = D(TT^*) \subset D(T^*)$

$\Rightarrow |\langle Ty, x\rangle| \leq \limsup |\langle y, T^*x_n\rangle| \leq$

$\leq \|y\| \cdot \limsup \|T^*x_n\| \overset{(1)}{=} \|y\| \limsup \|Tx_n\| =$

$= \|y\| \cdot \|Tx\|$

$\Rightarrow y \mapsto \langle Ty, x\rangle$ is cts on $D(T)$, i.e. $x \in D(T^*)$

Moreover, the computation shows that $\|T^*x\| \leq \|Tx\|$ $\lrcorner$

(3) As T^* is also normal, by (2) it follows $D(T^*) = D(T)$, $\|Tx\| = \|T^*x\|$, $x \in D(T)$, so (a), (b) is proved

(4) $S \supset T$, S normal $\Rightarrow S^* \subset T^*$

so $D(T) \subset D(S) = D(S^*) \subset D(T^*) = D(T)$

$\Rightarrow D(T) = D(S)$