VI.5 Complements to the theory of unbounded operators

Proposition 24. *Let E be an abstract spectral measure in a Hilbert space H defined on a σ-algebra $\mathcal{A}$. For an $\mathcal{A}$-measurable function $f : \mathbb{C} \to \mathbb{C}$ set $\Phi(f) = \int f \,\mathrm{d}E$.*

(1) *Let $x \in H$. Set $H_x = \{\Phi(f)x; f \in \mathbb{C}_b(\mathbb{C})\}$. Then H_x is a (not necessarily closed) subspace of H and the mapping $U_x : f \mapsto \Phi(f)x$ is a linear isometry of the space $L^2(E_{x,x})$ onto $\overline{H_x}$.*

(2) *There exists a set $\Gamma \subset S_H$ satisfying:*
   - *$H_x \perp H_y$ for $x, y \in \Gamma$, $x \neq y$.*
   - *$\operatorname{span}(\bigcup_{x\in\Gamma} H_x)$ is a dense subspace H.*

(3) *Let $\Omega = \Gamma \times \mathbb{C}$. Let*

$$\tilde{\mathcal{A}} = \{A \subset \Omega; \forall x \in \Gamma : \{\lambda \in \mathbb{C}; (x, \lambda) \in A\} \in \mathcal{A}\}$$

and

$$\mu(A) = \sum_{x\in\Gamma} E_{x,x}(\{\lambda \in \mathbb{C}; (x, \lambda) \in A\}), \quad A \in \tilde{\mathcal{A}}.$$

Then $(\Omega, \tilde{\mathcal{A}}, \mu)$ is a measure space (with a nonnegative measure). Moreover, the mapping $U : L^2(\mu) \to H$ defined by

$$U(g) = \sum_{x\in\Gamma} \Phi(\lambda \mapsto g(x, \lambda))x, \quad g \in L^2(\mu)$$

is a linear isometry of $L^2(\mu)$ onto H.

(4) *Let $f : \mathbb{C} \to \mathbb{C}$ be an $\mathcal{A}$-measurable function. Then $\Phi(f) = UM_{\tilde{f}}U^*$, where*

$$\tilde{f}(x, \lambda) = f(\lambda), \quad (x, \lambda) \in \Omega$$

anf $M_{\tilde{f}}$ is the operator on $L^2(\mu)$ given by

$$M_{\tilde{f}}g = \tilde{f} \cdot g, \quad g \in D(M_{\tilde{f}}) = \{g \in L^2(\mu); \tilde{f} \cdot g \in L^2(\mu)\}.$$

Theorem 25 (diagonalization of a normal operator). *Let T be a normal operator on a Hilbert space H. Then T is unitarily equivalent to a suitable multiplication operator. I.e., there exist a nonnegative measure μ, a unitary operator $U : L^2(\mu) \to H$ and a μ-measurable function f such that $T = UM_fU^*$, where M_f is defined as in Proposition 24. Moreover:*

(a) *If T is selfadjoint, f can be chosen to be real-valued.*
(b) *If T is bounded, f can be chosen to be bounded.*
(c) *If H is separable, μ can be chosen to be σ-finite.*

Theorem 26 (an alternative expression of the spectral decomposition of a selfadjoint operator). *Let T be a selfadjoint operator on H and let E be its spectral measure (from Theorem 36). Then $E(\mathbb{C} \setminus \mathbb{R}) = 0$. For $\lambda \in \mathbb{R}$ set $E_\lambda = E((-\infty, \lambda])$. Then:*

(a) *E_λ is an orthogonal projection for each $\lambda \in \mathbb{R}$.*
(b) *$E_\lambda E_\mu = E_\mu E_\lambda = E_{\min\{\lambda,\mu\}}$ for $\lambda, \mu \in \mathbb{R}$.*
(c) *$\lim\limits_{\mu\to\lambda+} E_\mu x = E_\lambda x$ for each $x \in H$ and $\lambda \in \mathbb{R}$.*
(d) *If λ is not an eigenvalue of T, then $\lim\limits_{\mu\to\lambda-} E_\mu x = E_\lambda x$ for each $x \in H$.*
(e) *If λ is an eigenvalue of T, then the formula $P_\lambda x = \lim\limits_{\mu\to\lambda-} E_\mu x$, $x \in H$, defines an orthogonal projectiom such that $E_\lambda - P_\lambda$ is also an orthogonal projection and, moreover, $R(E_\lambda - P_\lambda) = \operatorname{Ker}(\lambda I - T)$.*
(f) *$\lim\limits_{\mu\to-\infty} E_\mu x = 0$ and $\lim\limits_{\mu\to+\infty} E_\mu x = x$ for each $x \in H$.*
(g) *A real number λ belongs to $\rho(T)$ if and only if the mapping $\mu \mapsto E_\mu$ is constant on a neighborhood of λ.*

Theorem 27 (selfadjoint operators on a real Hilbert space). *Let H be a real Hilbert space and let T be an operator na H. Then T^* can be defined in the same way as in the complex case (see Section V.4). Let H_C be the hilbertian complexification of H, i.e., the space $H_C = H + iH = \{x + iy; x, y \in H\}$ equipped with the scalar product*

$$\langle x+iy, u+iv\rangle = \langle x,u\rangle + \langle y,v\rangle + i\langle y,u\rangle - i\langle x,v\rangle, \quad x+iy, u+iv \in H_C.$$

Define an operator T_C on H_C by

$$T_C(x+iy) = T(x) + iT(y), \quad x+iy \in D(T_C) = D(T) + iD(T).$$

Then:

(a) *If T is densely defined, then T_C is also densely defined and $(T_C)^* = (T^*)_C$.*
(b) *If $T \in L(H)$, then $T_C \in L(H_C)$ and $\|T_C\| = \|T\|$.*
(c) *If T is selfadjoint, then T_C is also selfadjoint and, moreover, for $\lambda \in \mathbb{R}$ we have*

$$\lambda I - T \text{ is invertible in } L(H) \Leftrightarrow \lambda I_C - T_C \text{ is invertible in } L(H_C).$$

(d) *Let T be selfadjoint and let E be the spectral measure of T_C, let $\mathcal{A}$ be the corresponding σ-algebra. Then the formula*

$$E_R(A) = E(A)|_H, \quad A \in \mathcal{A}$$

defines a "real spectral measure" on $\mathbb{R}$ and $T = \int \mathrm{id}\, \mathrm{d}E_R$.

Corollary 28. *Let H be a real Hilbert space.*

(i) *If $T \in L(H)$ is self-adjoint, then $\sigma(T)$ is a nonempty compact subset of $\mathbb{R}$. Moreover, given a real-valued continuous function f on $\sigma(T)$ we may define $\tilde{f}(T)$ (as the restriction of $\tilde{f}(T_C)$ to H). The assignment $f \mapsto \tilde{f}(T)$ then satisfies conditions (a)–(e) from Theorem IV.38 (after obvious adjustments).*
(ii) *If $T \in L(H)$, we may define the operator $|T| = \sqrt{T^*T}$ (as in Section V.1). Theorem V.8 is therefore valid for real space as well.*
(iii) *If $T \in L(H)$ is compact and self-adjoint, the statement of Theorem V.9 holds (the numbers λ_k are real). Further, Theorem V.10 holds for a real-valued function f.*
(iv) *If $T \in L(H)$ is compact, the statement of Theorem V.11 holds.*