VII.4 Weakly compact sets and operators in Banach spaces

Reminder, definitions and remarks: Let T be a Hausdorff completely regular space and $A \subset T$.

- The set A is said to be **compact** if any cover of A by open sets admits a finite subcover. Furhter, A is compact if and only if any net v A has a cluster point in A.
- Let $(x_\nu)_{\nu\in\Lambda}$ be a net in T. Recall that a point $x \in T$ is a cluster point of the net if for any neighborhood U of x and any $\nu_0 \in \Lambda$ there is $\nu \geq \nu_0$ with $x_\nu \in U$. Further,

$$\begin{aligned} x \text{ is a cluster point of } (x_\nu)_{\nu\in\Lambda} &\iff x \in \bigcap_{\nu_0\in\Lambda} \overline{\{x_\nu; \nu \geq \nu_0\}} \\ &\iff \text{there is a subnet of } (x_\nu)_{\nu\in\Lambda} \text{ converging to } x. \end{aligned}$$

- The set A is said to be **relatively compact** if its closure $\overline{A}$ is a compact subset of T. Furhter, A is relatively compact if and only if any net v A has a cluster point in T.
- A is said to be **countably compact** if any countable cover of A by open sets admits a finite subcover. Further, A is countably compact if and only if any sequence in A has a cluster point in A.
- Recall that a point x is a cluster point of the sequence (x_n) if any neighborhood of x contains x_n for infinitely many $n \in \mathbb{N}$. Further,

$$\begin{aligned} x \text{ is a cluster point of } (x_n) &\iff x \in \bigcap_{n_0\in\mathbb{N}} \overline{\{x_n; n \geq n_0\}} \\ &\iff \text{there is a subnet of the sequence } (x_n) \text{ converging to } x. \end{aligned}$$

- A is said to be **relatively countably compact**, if any sequence in A has a cluster point in T.
- A is said to be **sequentially compact** if any sequence in A has a subsequence converging to some element of A.
- A is said to be **relatively sequentially compact**, if any sequence in A has a subsequence converging to some element of T.

Remark: The following implications and no other ones hold among the notions defined above.

$$\begin{array}{ccccc} A \text{ compact} & \Rightarrow & A \text{ countably compact} & \Leftarrow & A \text{ sequentially compact} \\ \Downarrow & & \Downarrow & & \Downarrow \\ A \text{ relatively compact} & \Rightarrow & A \text{ relatively countably compact} & \Leftarrow & A \text{ relatively sequentially compact} \\ \Updownarrow & & \Uparrow & & \Uparrow \\ \overline{A} \text{ compact} & \Rightarrow & \overline{A} \text{ countably compact} & \Leftarrow & \overline{A} \text{ sequentially compact} \end{array}$$

In particular, the closure of a (relatively) countably compact set need not be countably compact and the closure of a (relatively) sequentially compact set need not be compact.

Remark: If T is a metric space and $A \subset T$, then

$$(*) \quad \begin{cases} A \text{ compact} \Leftrightarrow A \text{ countably compact} \Leftrightarrow A \text{ sequentially compact} \\ A \text{ relatively compact} \Leftrightarrow A \text{ relatively countably compact} \Leftrightarrow A \text{ relatively sequentially compact} \end{cases}$$

Definition. Let T be a Hausdorff completely regular topological space. T is said to be **angelic** if for any relatively countably compact subset $A \subset T$ the following assertions hold:

(i) A is relatively compact;
(ii) for each $x \in \overline{A}$ there exists a sequence (x_n) in A converging to x.

Remark: Any metric space is angelic.

Lemma 25. *Let T be an angelic space. Then the equivalences $(*)$ hold for any $A \subset T$.*

Theorem 26.

(a) *If K is a compact Hausdorff space, then the space $(\mathcal{C}(K), \tau_p)$ is angelic.*
(b) *If X is a Banach space, then the space (X, w) is angelic.*

Theorem 26 can be proved by combining the following three results.

Lemma 27. *Let K be a compact Hausdorff space and $A \subset \mathcal{C}(K)$ be τ_p-relatively countably compact. Then $\overline{A}^{\tau_p}$ is a τ_p-compact subset of $\mathcal{C}(K)$.*

Theorem 28 (Kaplansky). *Let K be a compact Hausdorff space, $f \in \mathcal{C}(K)$ and $A \subset \mathcal{C}(K)$. If $f \in \overline{A}^{\tau_p}$, then there is a countable set $C \subset A$ with $f \in \overline{C}^{\tau_p}$. (I.e., $(\mathcal{C}(K), \tau_p)$ has* **countable tightness**.*)*

Proposition 29. *Let K be a compact Hausdorff space and $A \subset \mathcal{C}(K)$. If (A, τ_p) is compact and separable, then it is metrizable.*

Theorem 30 (Eberlein-Šmulyan). *Let X be a Banach space and $A \subset X$. The following assertions are equivalent:*

(i) *A is relatively weakly compact.*
(ii) *A is relatively weakly countably compact.*
(iii) *A is relatively weakly sequentially compact.*

Similarly, the following assertions are equivalent:

(i') *A is weakly compact.*
(ii') *A is weakly countably compact.*
(iii') *A is weakly sequentially compact.*

Theorem 31 (Grothendieck). *Let K be a compact Hausdorff space and let $A \subset \mathcal{C}(K)$ be a bounded set.*

(a) *A is relatively weakly compact if and only if it is relatively τ_p-compact.*
(b) *A is weakly compact if and only if it is τ_p-compact.*

Definition. Let X and Y be Banach spaces and $T \in L(X, Y)$. An operator T is said to be **weakly compact** if $\overline{TB_X}$ is weakly compact.

Proposition 32. *Let X and Y be Banach spaces and $T \in L(X, Y)$.*

(a) *T is weakly compact if and only if for any bounded sequence (x_n) in X there exists a subsequence of the sequence (Tx_n) which is weakly convergent.*
(b) *If T is compact, then it is weakly compact.*
(c) *If at least one of the spaces X, Y is reflexive, then T is weakly compact.*

Theorem 33 (Gantmacher). *Let X and Y be Banach spaces and $T \in L(X, Y)$. The following assertions are equivalent:*

(i) *T is weakly compact.*
(ii) *The dual operator T' is weakly compact.*
(iii) *The dual operator T' is continuous from (Y^*, w^*) to (X^*, w).*
(iii) *$T''(X^{**}) \subset \varkappa(Y)$.*

Theorem 34 (Krein). *Let X be a Banach space and let $K \subset X$ be weakly compact. Then $\overline{\operatorname{aco} K}$ is weakly compact as well.*

Remark: The following nontrivial **James theorem** holds:

Let X be a Banach space and let $A \subset X$ be a weakly closed set (this is satisfied, e.g., if A is closed and convex). If for each $f \in X^*$ we have that $\operatorname{Re} f$ attains its maximum on A, then A is weakly compact.