V.2 The notion of an unbounded operator between Banach spaces

Definition. Let X and Y be Banach spaces over $\mathbb{F}$.

- By an **operator from** X **to** Y we mean a linear mapping $T : D(T) \to Y$, where $D(T)$ (the **domain** of the operator T) is a vector subspace of X.
- The range of the operator T, i.e. the set $T(D(T))$, is denoted by $R(T)$.
- An operator T from X to Y is called **densely defined**, if its domain $D(T)$ is dense in X.
- By the **graph of an operator** T we mean the set

$$G(T) = \{(x, y) \in X \times Y : x \in D(T) \ \& \ Tx = y\}.$$

- An operator T is said to be **closed** if its graph $G(T)$ is a closed subset of $X \times Y$, i.e., if for any sequence (x_n) in $D(T)$ satisfying
  - $x_n \to x$ for some $x \in X$,
  - $Tx_n \to y$ for some $y \in Y$;

  one has $x \in D(T)$ and $Tx = y$.
- Let S and T be operators from X to Y. We write $S \subset T$ if $G(S) \subset G(T)$; i.e., if $D(S) \subset D(T)$ and $Tx = Sx$ for each $x \in D(S)$. The operator T is then called an **extension** of the operator S.
- Let S and T be operators from X to Y. By their **sum** we mean the operator $S + T$ with domain $D(S + T) = D(S) \cap D(T)$ defined by the formula $(S + T)x = Sx + Tx$ for $x \in D(T + S)$.
- Let T be an operator from X to Y and $\alpha \in \mathbb{F}$. If $\alpha = 0$, by αT we mean the zero operator defined on X; if $\alpha \neq 0$, by αT we mean the operator defined by the formula $(\alpha T)x = \alpha \cdot Tx$ on $D(\alpha T) = D(T)$.
- Let T be an operator from X to Y, let S be an operator from Y to a Banach space Z. By their composition we mean the operator ST with domain

$$D(ST) = \{x \in D(T) : Tx \in D(S)\}$$

  defined by the formula $(ST)(x) = S(T(x))$ for $x \in D(ST)$.
- If T is a one-to-one operator from X to Y, by the **inverse operator of** T we mean the operator T^{-1} from Y to X, whose domain is $D(T^{-1}) = R(T)$ and which is the inverse mapping of T.

Examples 12.

(1) *Let $D(T) = \mathcal{C}^1([0, 1]) \subset\subset \mathcal{C}([0, 1])$ and let $T(f) = f'$ for $f \in D(T)$. Then T is a closed densely defined operator from $\mathcal{C}([0, 1])$ to $\mathcal{C}([0, 1])$.*

(2) *Let $D(U) = \{f \in \mathcal{C}^1([0, 1]); f'(0) = 0\} \subset\subset \mathcal{C}([0, 1])$ and let $U(f) = f'$ for $f \in D(U)$. Then U is a closed densely defined operator from $\mathcal{C}([0, 1])$ to $\mathcal{C}([0, 1])$ and, moreover, $U \subsetneqq T$, where T is the operator from (1).*

(3) *Let $D(S)$ be the subspace $\mathcal{C}([0, 1])$ consisting of all the polynomials and let $S(f) = f'$ for $f \in D(S)$. Then T is a densely defined operator from $\mathcal{C}([0, 1])$ to $\mathcal{C}([0, 1])$, which is not closed, but has a closed extension (the operator T from (1)).*

(4) *Let $D(T)$ be a subspace of ℓ^2 made by the vector with finitely many nonzero coordinates. For $x = (x_n) \in D(T)$ set $Tx = (\sum_{n=1}^{\infty} x_n, 0, 0, \dots)$. Then T is a densely defined operator from ℓ^2 to ℓ^2, which has no closed extension.*

Lemma 13 (on the graph of an operator). *A subset $L \subset X \times Y$ is the graph of an operator from X to Y if and only if it is a linear subspace satisfying*

$$\{(x, y) \in L : x = 0\} = \{(0, 0)\}.$$

Proposition 14. *For operators R, S, T between Banach spaces (for which the given operations are defined) one has:*

(i) $(R+S)+T = R+(S+T)$;
(ii) $(RS)T = R(ST)$;
(iii) $(R+S)T = RT+ST$ *and* $T(R+S) \supset TR+TS$. *If T is everywhere defined, then* $T(R+S) = TR+TS$.

Proposition 15 (on closed operators). *Let T be an operator from X to Y.*

(a) *If T is closed and $D(T) = X$, then $T \in L(X,Y)$.*
(b) *T has a closed extension if and only if $(x_n, Tx_n) \to (0, y)$ in $D(T) \times Y$ implies $y = 0$.*
(c) *If T is closed and one-to-one, then T^{-1} is closed as well.*

Notation. If T is an operator from X to Y, which has a closed extension, by the symbol $\overline{T}$ we denote its minimal closed extension, i.e., the operator whose graph $G(\overline{T})$ is $\overline{G(T)}$, the closure of the graph of T in $X \times Y$.

Proposition 16. *Let T be a closed operator from X to Y. Then:*

(a) *If $S \in L(X,Y)$, then $S+T$ is a closed operator and $D(S+T) = D(T)$.*
(b) *If $S \in L(Y,Z)$, then $D(ST) = D(T)$. If S is, moreover, an isomorphism of Y into Z, then ST is closed.*
(c) *If $S \in L(Z,X)$, then TS is closed.*

Examples 17.

(1) *Let $X = \mathcal{C}([0,1])$, $D(T) = \mathcal{C}^1([0,1])$, $T(f) = f'$ for $f \in D(T)$ and $Sf = \sum_{n=1}^{\infty} \frac{1}{2^n} f(\frac{1}{n})$ for $f \in \mathcal{C}([0,1])$ (the result is a constant function). Then T is densely defined and closed, $S \in L(X)$, but ST has no closed extension.*
(2) *Let $X = \ell^2$, $Y = \{(x_n) \in \ell^2; \sum_{n=1}^{\infty} |nx_n|^2 < \infty\}$. For $(x_n) \in Y$ set*

$$T((x_n)) = (0, x_1, 2x_2, 3x_3, \dots),$$
$$S((x_n)) = (\sum_{n=1}^{\infty} x_n, -x_1, -2x_2, -3x_3, \dots).$$

Then S and T are densely defined closed operators, but $S+T$ has no closed extension.

Proposition 18 (on the inverse to a closed operator). *Let T be a one-to-one closed operator from X to Y. The following assertions are equivalent:*

(i) *$R(T) = Y$ and $T^{-1} \in L(Y,X)$.*
(ii) *$R(T) = Y$.*
(iii) *$R(T)$ is dense in Y and T^{-1} is continuous on $R(T)$.*

Remark. For non-closed operators the assertions from the previous proposition are not equivalent. More precisely: If T is an operator from X to Y, which is not closed, then:

- The assertion (i) cannot hold.
- The assertion (ii) may hold. If it holds, then neither (i) nor (iii) hold. In this case T may or may not have a closed extension. If it has a closed extension, then the operator $\overline{T}$ is not one-to-one.
- The assertion (iii) may hold. If it holds, then neither (i) nor (ii) hold.In this case T may or may not have a closed extension. If it has a closed extension, then the operator $\overline{T}$ satisfies the equivalent conditions from the previous proposition.